Study of frames in Hilbert C^*-modules and their representation as regular k-distance sets

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Declaration

I hereby declare that

i) the thesis comprises of my original work towards the degree of Doctor of Philosophy at Dhirubhai Ambani Institute of Information and Communication Technology and has not been submitted elsewhere for a degree,

ii) due acknowledgment has been made in the text to all the reference material used.

Ekta Rajput

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Certificate

This is to certify that the thesis work entitled STUDY OF FRAMES IN HILBERT C^*-MODULES AND THEIR REPRESENTATION AS REGULAR K-DISTANCE SETS has been carried out by EKTA RAJPUT for the degree of Doctor of Philosophy at *Dhirubhai Ambani Institute of Information and Communication Technology* under my supervision.

NKsahu

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Vande Vishnum Bhava-Bhaya-Haram Sarva-Loka-Eka-Naatham ||

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Contents

Abstract

The definition of basis in the study of vector space is very antagonistic. As a result of that, one might look for a prominent substitute. Frames are such a notion, as the linear independence between the frame elements is not required. Further, the additional degree of freedom coming from the structure of C^*-algebra $\mathcal{A}$ enriches the theory of frames in Hilbert C^*-modules. This thesis aims to introduce various notions of frame theory in Hilbert C^*-modules as they are the subjects of the recent study. We also introduced the notion of a regular k-distance frame in Hilbert space. The thesis is planned to be organized into six chapters, along with the introductory and literature survey chapter and a chapter for conclusions and the scope of future research. Chapter 1 of the thesis is the introductory chapter, where a brief introduction of frame theory in Hilbert space as well as in Hilbert C^*-module has been discussed. The interest in taking the particular research problem has been outlined. A concise but sufficient literature survey has been presented. In Chapter 2, we introduced the concept of a regular k-distance frame in Hilbert space as well as focused on k-distance tight frames for the underlying space. We have introduced the definition of dual frames for a regular k-distance set. Finally, the perturbation result for regular k-distance frames is established. The objective of Chapter 3 is to introduce woven g-frames in Hilbert C^*-modules and to develop its fundamental properties. This study establishes sufficient conditions under which two g-frames possess weaving properties. We also investigated the sufficient conditions under which a family of g-frames includes weaving properties. Chapter 4 is concerned with weaving K-frames in Hilbert C^*-module. We introduced the concept of weaving K-frames and defined an atomic system for weaving K-frames in Hilbert C^*-module. We studied weaving K-frames in this chapter from the operator

theoretic approach. Moreover, we gave an equivalent definition for weaving K-frames. In Chapter 5, we introduced the notion of a controlled K-frame in Hilbert C^*-modules. We established the equivalent condition for controlled K-frame in Hilbert C^*-modules. We investigated some operator theoretic characterizations of controlled K-frames and controlled Bessel sequences. Moreover, we established the relationship between the K-frames and controlled K-frames. We also investigated the invariance of a C-controlled K-frame under a suitable map T. At last, we proved a perturbation result for controlled K-frame in Hilbert C^*-modules.
An equivalent definition is much easier to apply and permits us to study the various types of frames from the operator theory point of view. The multiple notions of frame theory developed in this thesis will draw the attention of researchers to work in this area. At last, in Chapter 6, we summarize all the work that has been done so far and feature the potential avenues for the future scope of research.

Motivation and Objective of the Thesis

In a vector space, a set of vectors is referred to as a basis if every element in the underlying space can be expressed in terms of a finite linear combination of the basis vectors uniquely. The definition of basis in the study of vector space is very antagonistic. As a result of that, one might look for a prominent substitute. Frames are such a notion as the linear independence between the frame elements is not required. In addition to that, the additional degree of freedom coming from the structure of C^*-algebra $\mathcal{A}$ enriches the theory of frames in Hilbert C^*-modules.
We intend to see whether the results of frame theory in Hilbert spaces hold for frame theory in Hilbert C^*-modules and, if not, then to study what modifications we need. In **Chapter 2**, we investigated the concept of a regular k-distance frame in Hilbert space which is the extension of a regular two-distance frame in Hilbert space. A regular two-distance frame is a particular type category of frame which has some nice properties. Motivated by this, we studied regular k-distance frames, in particular, regular tight k-distance frames in Hilbert space. Tight frames are those in which the lower and upper frame bounds are equal. Tight frames play a key role in wide applications as tight frames look like a more natural way to reconstruct

vectors. Tight frames are closest to orthonormal bases as they are a redundant set of vectors and have properties like basis. In **Chapter 3** and **Chapter 4**, we studied the concept of woven frames in Hilbert C^*-modules. Recently many people got significant results in frame theory by generalizing the results which are present in Hilbert space to Hilbert C^*-modules. The concept of weaving frames is applicable in wireless sensor networks that require distributed processing under different frames, as well as pre-processing of signals using Gabor frames. Generalized frames (or g-frames) include standard frames, bounded invertible linear operators, and many recent generalizations of frames. g-frames in Hilbert C^*-modules interest many useful properties with their comparable tools in Hilbert space. As we know, K-frames and standard frames diverge in many aspects; we introduce the concept of weaving K-frames and define an atomic system for weaving K-frames in Hilbert C^*-modules. As it is easier to work, we gave an equivalent definition for weaving K-frames and characterized weaving K-frames from the operator theory point of view. In **Chapter 5**, we introduced the notion of controlled K-frames in Hilbert C^*-modules. Controlled frames have been an area of interest because of their expertise in improving the numerical efficiency of iterative algorithms for inverting the frame operator.

List of Principal Symbols and Acronyms

$\mathbb{N}$	Set of natural numbers
$\mathbb{R}^n$	n-dimensional real space
i, j	Indexes
I, J	Index set
$\mathbb{I}$	Identity matrix
$\mathbb{J}$	Matrix whose all entries are 1's
H	Hilbert space
$\mathcal{H}$	Hilbert C^*-module over unital C^*-algebra $\mathcal{A}$
$L(\mathcal{H})$	The set of all adjointable operators on Hilbert C^*-module $\mathcal{H}$
$GL^+(\mathcal{H})$	The set of all bounded linear positive invertible operators on $\mathcal{H}$ with bounded inverse
$[m]$	The set $\{1, 2, ..., m\}$, where m is any natural number
$\square$	End of the proof

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CHAPTER 1

Introduction

For a Hilbert space H, the Plancherel equality states that

$$\sum_k |\langle f, e_k \rangle| = \|f\|^2, \; \forall\, f \in H. \tag{1.1}$$

Each orthonormal basis $\{e_k\}$ for a Hilbert space H satisfies the Plancherel equality. However, to satisfy the Plancherel equality, a sequence may not be orthonormal or a basis. A sequence that satisfies the Plancherel equality is called a Parseval frame; the definition of a frame in Hilbert space H enforces a weak requirement. Also, the basis is one of the important concepts in the field of vector spaces. The set of vectors $\{f_k\}_{k=1}^{\infty}$ in a Hilbert space H forms a basis if $\{f_k\}_{k=1}^{\infty}$ spans H and also linearly independent. Every $f \in H$ can be represented as

$$f = \sum_{k=1}^{\infty} c_k f_k. \tag{1.2}$$

The coefficient c_k are uniquely determined.

The condition of linear independence is very restrictive; thus, one might look for an alternative tool. Frames are such a notion that provides liberty in linear independence. A frame also allows every vector in the space to be written as in equation (1.2), but the corresponding coefficients are not necessarily unique.

We begin with the definition of frame in Hilbert space H.

Definition 1.1. *A sequence $\{f_k\}_{k=1}^{\infty}$ of elements in Hilbert space H is a frame for H if* *there exist constants* $A, B > 0$ *such that*

$$A\|f\|^2 \le \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \le B\|f\|^2, \ \forall\, f \in H. \tag{1.3}$$

The constants A and B are called *lower* and *upper frame bounds*, respectively. The optimal upper frame bound is the infimum over all upper frame bounds and the optimal lower frame bound is the supremum over all lower frame bounds.

Gabor [32] carried out a method in 1946 to analyze the information conveyed and its transmission by different communication channels such as speech, telegraphy, telephony, radio, or television. In 1952, Duffin and Schaeffer [23] formally introduced frames in Hilbert spaces while studying the non-harmonic Fourier series. Duffin and Schaeffer extracted Gabor's method to define the notion of frames for a Hilbert space. As the linear independence between the frame elements is not required, they can be viewed as more relaxed substitutes of bases in Hilbert spaces. The subject does not grab the attention of people for quite some time. It took almost three decades to realize the potential of the frame theory. In 1980, Robert M. Young [56] wrote an introductory book entirely devoted to the non-harmonic Fourier series. As the wavelet era began in 1985, Daubechies, Grossmann, and Meyer [22] reintroduced and developed the theory of frames in 1986. After this revolutionary work, frame theory started getting the attention of the community because it plays as a central tool in many applied areas like signal processing [30], coding and communications [52], image processing [11], time-frequency analysis, sampling theory[24, 25], data compression, numerical analysis, wavelet theory, filter theory [10]. Moment, indeed more applications of the frame theory are being found, such as signal detection, compressive sensing, data analysis, optics, and numerous other areas.

Frame theory still has plenty of open fundamental problems from various advanced fields, like Gabor frames or Weyl-Heisenberg frames, related to a dynamical sampling in a Hilbert space H. The problem of finding good estimates for the lower frame bound for a finite collection of exponentials in $L^2(-\pi, \pi)$.

Fusion frame is a generalization of frames that Cassaza and Kutyniok [14] introduced in 2003 and investigated in [2, 15, 44, 46]. The purpose of introducing a fusion frame or frame of subspace comes from signal processing, to be more specific, the ambition of accurately processing and analyzing huge data sets. Fusion frames have wide applications in distributed sensing, parallel processing, packet encoding, and so on. Perturbation theorems for frames in Hilbert space are essential and valuable tools to construct new frames near the given one. Over the last decade, numerous researchers have generalized the Paley-Wiener perturbation theorem to the perturbation of frames in Hilbert spaces (see [3], [18], [20], [19]). Casazza and Christensen [18] attained the most general result of these.

Equiangular tight frames (ETFs) are helpful for signal reconstruction when all the phase information is lost [8]. Strohmer and Heath [52] and Holmes and Paulsen [39] initiated the study of equiangular tight frames. In particular, [39] studied frames from the viewpoint of coding theory and found that equiangular tight frames give error-correcting codes that are robust against two erasures. Latterly equiangular tight frames (ETFs) are generalized to two-distance tight frames. Equiangular tight frames (ETFs) are an essential class of finite-dimensional frames. Barg *et al.* [5] in 2015 characterized finite tight frames, which are also of two-distance sets. Finite tight frames are the most spontaneous generalization of orthonormal bases. In [17], authors deeply study regular two-distance sets. They presented various properties of these sets as well as focused on the case where they form tight frames for the underlying space. They discussed some constructions of regular two-distance sets, in particular, two-distance tight frames. Tight frames play a crucial part in wide operations as they look like a more natural way to reconstruct vectors. Tight frames are closest to orthonormal bases as they are a spare set of vectors and have properties like basis.

Feichtinger and Werther [29] presented a family of analysis and synthesis systems with frame-like properties for the closed subspaces of a separable Hilbert space. The motivation for the definition of atomic system (or local atoms) is based on examples from sampling theory [28]. The atomic system (or local atoms) is capable of generating a proper subspace, although they do not belong to them. Li and Ogawa [48] proposed the family of local atoms for a closed subspace of Hilbert space called pseudo-frame. In 2012, L.Găvruta [33] introduced the notion of K-

frames in Hilbert space to study the atomic systems with respect to a bounded linear operator K, which advance the results of Feichtinger and Werther. In this, a generalization of frames is discussed, which allows to reconstruct elements from the range of a linear and bounded operator in a Hilbert space.

Controlled frames in Hilbert spaces have been introduced by P.Balazs [4] to improve the numerical effectiveness of iterative algorithms for flipping the frame operator. In 2005, controlled frames were used as a tool for spherical wavelets [9]. In 2016, Hua and Huang [40] introduced a controlled $K-g$-frame and proposed several methods to construct controlled $K-g$-frames. In 1977, Coifman and Weiss [21] introduced the concept of atomic decomposition for function spaces, the H^p spaces. After that, Feichtinger and Gröchenig [27] extended this idea to Banach spaces. Gröchenig [35] introduced a more general notion of Banach frames for Banach spaces and called them atomic decompositions. Casazza, Han, and Larson [13] also studied atomic decompositions and Banach frames for Banach spaces. Lately, several generalizations of frames in Banach spaces have been introduced and studied. Han and Larson [38] defined a Schauder frame for a Banach space E to be an inner direct summand (i.e., a compression) of a Schauder basis of E. Frame theory plays a vital part in the study of Besov spaces in Banach's space theory.

In recent times, numerous mathematicians got significant results by generalizing the frame theory in Hilbert spaces to frame theory in Hilbert C^*-modules which enrich the theory of frames. Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of real or complex numbers. They were introduced and investigated originally by Kaplansky [42]. Frank and Larson [31] defined the concept of standard frames in finitely or countably generated Hilbert C^*-modules over a unital C^*-algebra. For further details of frames in Hilbert C^*-modules, one may relate to Doctoral Dissertation [41], Han et al. [37] and Han *et al.* [36].

In [31], authors proved that every countably generated Hilbert module over a unital C^*-algebra admits frames by using Kasparov's stabilization theorem.

Paschke [50] defined pre-Hilbert B-modules without the restriction that B be commutative in an analogous way as I. Kaplansky's "C^*-modules" [43]. Later, Kasparov's stabilization theorem and work done by Paschke in [50] turn out to be a provocation for frames in Hilbert C^*-modules. In [49], Najati *et al.* introduced the concepts of the atomic system for operators and K-frames in Hilbert C^*-modules. Rashidi and Rahimi [51] introduced controlled frames in Hilbert C^*-modules and showed that they share various beneficial properties with their corresponding notions in a Hilbert space. The notion of weaving frames in Hilbert space was introduced in [6] and investigated in [12, 16]. The concept of weaving frames is partially motivated by preprocessing of Gabor frames and has potential applications in wireless sensor networks that require distributed processing under different frames, as well as preprocessing of signals using Gabor frames. In 2018, Deepshikha and Lalit K. Vashisht [55] studied the weaving properties of K-frames in Hilbert space. They presented necessary and sufficient conditions for weaving K-frames in Hilbert spaces and sufficient conditions for Paley–Wiener type perturbation of weaving K-frames. Also, it is shown that woven K-frames and weakly woven K-frames are equivalent. Woven frames for finitely or countably generated Hilbert C^*-module were introduced and studied in [34]. Authors have investigated some properties of woven frames and obtained some conditions on a perturbed family of sequences. In [45], Khosravi introduced fusion frames and g-frames in Hilbert C^* modules and showed that they share many useful properties with their corresponding notions in Hilbert space. They also generalized a perturbation result in frame theory to g-frames in Hilbert spaces. In [45], fusion frames in Hilbert C^*-modules were introduced, and authors showed that they share many beneficial properties with their corresponding notions in Hilbert space.

Chapter 1 of the thesis is the introductory chapter, where a brief introduction of frame theory in Hilbert space as well as in Hilbert C^*-module have been discussed. The interest in taking the particular research problem has been outlined. A concise but sufficient literature survey has been discussed.

In **Chapter 2**, we introduced the concept of regular k-distance frame in Hilbert space. Additionally, we discussed various characteristics of regular k-distance frames and focused on k-distance tight frames for the underlying space. We have introduced the definition of dual frames for regular k-distance set and in support presented example also. Finally, the perturbation result for regular k-distance frames is established. When k-distance sets form frames and tight frames for the space, we call them k-distance frames and k-distance tight frames, respectively.

The objective of **Chapter 3** is to introduce woven g-frames in Hilbert C^*-modules, and to develop its fundamental properties. Woven frames are widely applicable in wireless sensor networks and are induced by a problem in distributed signal processing. g-frames provide more choices for analyzing functions from the frame expansion coefficients. This study establishes sufficient conditions under which two g-frames possess the weaving properties. We also investigated the sufficient conditions under which a family of g-frames possesses weaving properties.

Chapter 4 is concerned with weaving K-frames in Hilbert C^*-module. As K-frames and standard frames diverge in many aspects, we introduced the concept of weaving K-frames and defined an atomic system for weaving K-frames in Hilbert C^*-module. In this chapter, we studied weaving K-frames from the operator theoretic approach. Moreover, we gave an equivalent definition for weaving K-frames and characterized weaving K-frames in terms of bounded linear operators. We also studied that woven Bessel sequences are invariant under an adjointable operator.

In **Chapter 5** we introduced the notion of controlled K-frame in Hilbert C^*-modules. Controlled frames have been the subject of interest because of their ability to improve the numerical efficiency of iterative algorithms for inverting the frame operator. We established the equivalent condition for controlled K-frame in Hilbert C^*-modules. We investigate some operator theoretic characterizations of controlled K-frames and controlled Bessel sequences. Moreover, we established the relationship between the K-frames and controlled K-frames. We also investigated the invariance of a C-controlled K-frame under a suitable map T. At last, we proved a perturbation result for controlled K-frame in Hilbert C^*-modules.

Finally, in **Chapter 6**, we summarize all the work that has been done so far and feature the potential avenues for the future scope of research.

1.1 Frames in Hilbert space

First we recall the definition of frame in Hilbert space H.

Definition 1.2. *A sequence $\{f_k\}_{k=1}^{\infty}$ of elements in Hilbert space H is a frame for H if there exist constants $A, B > 0$ such that*

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \ \forall\, f \in H. \tag{1.4}$$

The constants A and B are called *lower and upper frame bounds*, respectively.

Definition 1.3. *A sequence $\{f_k\}_{k=1}^{\infty}$ of elements in Hilbert space H is called a Bessel sequence for H if there exists a real constant $B > 0$ such that*

$$\sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \ \forall\, f \in H. \tag{1.5}$$

B is referred as Bessel bound for the Bessel sequence $\{f_k\}_{k=1}^{\infty}$.

Definition 1.4. *Assume $\{f_k\}_{k=1}^{\infty}$ is a frame for a Hilbert space H.*

1. *If $A = B$, then $\{f_k\}_{k=1}^{\infty}$ is called a tight frame, to be precise we say that $\{f_k\}_{k=1}^{\infty}$ is an A-tight frame.*
2. *If $A = B = 1$, $\{f_k\}_{k=1}^{\infty}$ is called a Parseval frame.*
3. *If $\{f_k\}_{k=1}^{\infty}$ ceases to be a frame whenever any single element is deleted from the sequence, then $\{f_k\}_{k=1}^{\infty}$ is called an exact frame.*

One fruitful approach to frame theory for infinite-dimensional Hilbert spaces is to study frames in an operator theoretic approach. Suppose that $\{f_k\}_{k=1}^{\infty}$ is a frame for a Hilbert space H. Then we define the following operators.
The operator $T\colon H \to \ell^2$ is defined by

$$Tf = \{\langle f, f_k\rangle\}_{k=1}^{\infty}$$

is called the *analysis operator*.

The adjoint operator $T^*\colon \ell^2 \to H$ is obtained as

$$T^*\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k f_k.$$

T^* is called *pre-frame operator or the synthesis operator*.

By composing T and T^*, we obtain the *frame operator* $S\colon H \to H$

$$Sf = T^*Tf = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k. \tag{1.6}$$

Now we give an example of a frame in Hilbert Space.

Example 1.1. *Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for a Hilbert space H.*

(1) By repeating each element of $\{e_k\}_{k=1}^{\infty}$ thrice we have

$$\{f_k\}_{k=1}^{\infty} = \{e_1, e_1, e_1, e_2, e_2, e_2, e_3, e_3, e_3, \cdots\}$$

which is a tight frame with frame bound $A = B = 3$.

(2) Let

$$\{f_k\}_{k=1}^{\infty} = \{e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \cdots\}$$

For each $f \in H$, we have

$$\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 = \sum_{k=1}^{\infty} k |\langle f, \frac{1}{\sqrt{k}} e_k \rangle|^2 = \|f\|^2.$$

Therefore, $\{f_k\}_{k=1}^{\infty}$ is a Parseval frame.

Let's state some of the important properties of S:

Proposition 1.1. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame with frame operator S and frame bounds A, B. Then the following holds:*

1. *S is bounded, invertible, self-adjoint, and positive.*

2. *$\{S^{-1}f_k\}_{k=1}^{\infty}$ is a frame with bounds B^{-1}, A^{-1}. The frame operator for $\{S^{-1}f_k\}_{k=1}^{\infty}$ is S^{-1}.*

Now we give a reconstruction formula for any vector f in Hilbert space H using the inverse frame operator. A direct calculation yields that

$$f = SS^{-1}f = \sum_{k=1}^{\infty}\langle S^{-1}f, f_k\rangle f_k = \sum_{k=1}^{\infty}\langle f, S^{-1}f_k\rangle f_k = \sum_{k=1}^{\infty}\langle f, S^{-1/2}f_k\rangle S^{-1/2}f_k. \quad (1.7)$$

$\{\langle S^{-1}f, f_k\rangle\}_{k=1}^{\infty}$ are called the frame coefficients for $f \in H$ and $S^{-1/2}$ denote the positive square root of inverse frame operator S^{-1}.
Dual frames play an essential role in the reconstruction of vectors (or signals) in terms of frame elements.

Definition 1.5. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame of a Hilbert space H. We call a sequence $\{g_k\}_{k=1}^{\infty} \subseteq H$ a dual frame of $\{f_k\}_{k=1}^{\infty}$ if*

$$f = \sum_{k=1}^{\infty}\langle f, g_k\rangle f_k$$

holds true for every $f \in H$. In particular, $\{S^{-1}f_k\}_{k=1}^{\infty}$ is called the canonical dual (or standard dual) of $\{f_k\}_{k=1}^{\infty}$, where S is the frame operator of $\{f_k\}_{k=1}^{\infty}$.

We now list some characterizations of frames in Hilbert spaces from the operator theory point of view.

Theorem 1.1. *([41]) A sequence $\{f_k\}_{k=1}^{\infty}$ in Hilbert space H is a frame for H if and only if*

$$T\colon \{c_k\}_{k=1}^{\infty} \to \sum_{k=1}^{\infty} c_k f_k$$

is a well-defined mapping of ℓ^2 onto H.

Theorem 1.2. *([41]) A sequence $\{f_k\}_{k=1}^{\infty}$ in Hilbert space H is a frame for H with bounds A, B if and only if the following conditions are satisfied:*

1. *$\overline{span\{f_k\}_{k=1}^{\infty}} = H$;*

2. *The pre-frame operator T is well defined on ℓ^2 and*

$$A\sum_{k=1}^{\infty}|c_k|^2 \le \|T\{c_k\}_{k=1}^{\infty}\| \le B\sum_{k=1}^{\infty}|c_k|^2,\ \forall\ \{c_k\}_{k=1}^{\infty} \in (KerT)^{\perp}.$$

1.2 Frames in Hilbert C^*-modules

First we recall the notion of C^*-algebra and Hilbert C^* module.

Definition 1.6. *A $*$-algebra $\mathcal{A}$ is an algebra with a $*$-structure.*

$$*: \mathcal{A} \to \mathcal{A}$$

1. $(a+b)^* = a^* + b^*$
2. $(ab)^* = b^* a^*$
3. $(\alpha a)^* = \overline{\alpha} a^*$
4. $(a^*)^* = a$ *(Involution), for all $a, b \in \mathcal{A}$ and any scalar $\alpha \in \mathbb{C}$.*

Definition 1.7. *A C^*-algebra is a $*$-unital subalgebra of $B(\mathcal{H})$.*

Definition 1.8. *Let $\mathcal{A}$ be a C^*-algebra. An inner product $\mathcal{A}$-module is a complex vector space $\mathcal{H}$ such that*
(i) $\mathcal{H}$ is a right $\mathcal{A}$-module i.e there is a bilinear map

$$\mathcal{H} \times \mathcal{A} \to \mathcal{A}: (x, a) \to x \cdot a$$

satisfying $(x \cdot a) \cdot b = x \cdot (ab)$ and $(\lambda x) \cdot a = x \cdot (\lambda a)$, and $x \cdot 1 = x$, where $\mathcal{A}$ has a unit 1.
(ii) There is a map $\mathcal{H} \times \mathcal{H} \to \mathcal{A}: (x, y) \to \langle x, y \rangle$ satisfying
1. $\langle x, x \rangle \geq 0$
2. $\langle x, y \rangle^* = \langle y, x \rangle$
3. $\langle ax, y \rangle = a \langle x, y \rangle$
4. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
5. $\langle x, x \rangle = 0$ *if and only if* $x = 0$ *(for every $x, y, z \in \mathcal{H}$, $a \in \mathcal{A}$).*

Definition 1.9. *A Hilbert C^*-module over $\mathcal{A}$ is an inner product $\mathcal{A}$-module with the property that $(\mathcal{H}, \| \cdot \|_{\mathcal{H}})$ is complete with respect to the norm $\|x\| = \| \langle x, x \rangle \|_{\mathcal{A}}^{\frac{1}{2}}$, where $\|.\|_{\mathcal{A}}$ denotes the norm on $\mathcal{A}$.*

Note that in Hilbert C*-modules the Cauchy-Schwartz inequality is valid.

Proposition 1.2. *Let $\mathcal{H}$ be a Hilbert C^*-module, and $x, y \in \mathcal{H}$, then*

$$\|\langle x, y\rangle\|^2 \leq \|\langle x, x\rangle\| \cdot \|\langle y, y\rangle\|.$$

Let $\mathcal{A}$ be a C^*-algebra and consider

$$\ell^2(\mathcal{A}) = \{\{a_j\}_{j\in J} \in \mathcal{A} : \sum_{j\in J} a_j a_j^* \text{ converges in } \|\cdot\|_{\mathcal{A}}\}.$$

It is easy to see that $\ell^2(\mathcal{A})$ is a Hilbert C^*-module with point wise operations with the inner product and norm defined as

$$\langle\{a_j\}, \{b_j\}\rangle_{j\in J} = \sum_{j\in J} a_j b_j^*, \quad \{a_j\}, \{b_j\} \in \ell^2(\mathcal{A})$$

and

$$\|\{a_j\}\|_{j\in J} = \sqrt{\|\sum_{j\in J} a_j a_j^*\|}.$$

Definition 1.10. *([41]) Let $\mathcal{A}$ be a unital C^*-algebra and $j \in J$ be a finite or countable index set. A sequence $\{\psi_j\}_{j\in J}$ of elements in a Hilbert $\mathcal{A}$ -module $\mathcal{H}$ is said to be a frame if there exist two constants $C, D > 0$ such that*

$$C\langle f, f\rangle \leq \sum_{j\in J}\langle f, \psi_j\rangle\langle\psi_j, f\rangle \leq D\langle f, f\rangle,\ \forall\, f \in \mathcal{H}. \tag{1.8}$$

The frame $\{\psi_j\}_{j\in J}$ is said to be a *tight frame* if $C = D$, and is said to be *Parseval or a normalized tight frame* if $C = D = 1$.

Suppose that $\{\psi_j\}_{j\in J}$ is a frame of a finitely or countably generated Hilbert C^*-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. The operator $T\colon \mathcal{H} \to \ell^2(\mathcal{A})$ defined by

$$Tf = \{\langle f, \psi_j\rangle\}_{j\in J}$$

is called the *analysis operator*.

The adjoint operator $T^*\colon \ell^2(\mathcal{A}) \to \mathcal{H}$ is given by

$$T^*\{c_j\}_{j\in J} = \sum_{j\in J} c_j \psi_j.$$

T^* is called *pre-frame operator or the synthesis operator*.

By composing T and T^*, we obtain the *frame operator* $S\colon \mathcal{H} \to \mathcal{H}$

$$Sf = T^*Tf = \sum_{j\in J}\langle f, \psi_j\rangle\psi_j. \tag{1.9}$$

Theorem 1.3. *([31]) Let $\mathcal{A}$ be a unital C^*-algebra, $\mathcal{H}$ be a finitely or countably generated Hilbert $\mathcal{A}$-module and $\{x_j\}_{j\in J}$ be a Parseval frame (not necessarily standard orthonormal basis) of $\mathcal{H}$. Then the reconstruction formula*

$$x = \sum_{j\in J}\langle x, x_j\rangle x_j$$

holds for every $x \in \mathcal{H}$ in the sense of convergence with respect to the topology that is induced by the set of semi-norms $\{|f(\langle\cdot,\cdot\rangle)|^{1/2}\colon f \in \mathcal{A}^*\}$. The sum converges always in norm if and only if the frame $\{x_j\}_{j\in J}$ is standard.

We now give an example of a frame in the Hilbert C^*-module.

Example 1.2. *Let ℓ^∞ be the unitary C^*-algebra of all bounded complex-valued sequences with the following operations*

$$uv = \{u_jv_j\}_{j\in J},\ u^* = \{\overline{u_j}\}_{j\in j},\ ,\|u\| = \max_{j\in J}|u_i|,\ \forall\ u = \{u_j\}_{j\in J},\ v = \{v_j\}_{j\in J} \in \ell^\infty$$

Let $\mathcal{H} = C_0$ be the set of all sequences converging to zero. Then C_0 is a Hilbert ℓ^∞-module with ℓ^∞-valued inner product

$$\langle u, v\rangle = uv^* = \{u_jv_j^*\}_{j\in J} = \{u_j\overline{v_j}\}_{j\in J}\ \forall\ u, v \in C_0.$$

We define $\{x_j\}_{j\in J} \in C_0$ as follows:

$$\{x_j\}_{j\in J} = \{e_1, e_2, e_3, e_4, e_5, ...\},$$

where $\{e_j\}_{j\in J}$ be the standard orthonormal basis for $\mathcal{H}$.

Let $x = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, ...\} \in \mathcal{H}$. Then $\langle x, x\rangle = \{\alpha_1\alpha_1^*, \alpha_2\alpha_2^*, \alpha_3\alpha_3^*, \alpha_4\alpha_4^*, ...\}$. Here partial ordering $'\leq'$ means pointwise comparision.

Now, for the upper bound, we have

$$\begin{aligned}
\sum_{j\in J}\langle x,x_j\rangle\langle x_j,x\rangle &= \langle x,e_1\rangle\langle e_1,x\rangle+\langle x,e_2\rangle\langle e_2,x\rangle+\langle x,e_3\rangle\langle e_3,x\rangle+...\\
&= \{\alpha_1\alpha_1^*,0,0,0,0,...\}+\{0,\alpha_2\alpha_2^*,0,0,0,...\}+\{0,0,\alpha_3\alpha_3^*,0,0,...\}+...\\
&= \{\alpha_1\alpha_1^*,\alpha_2\alpha_2^*,\alpha_3\alpha_3^*,...\}\\
&= \sum_{j\in J}\langle x,e_j\rangle\langle e_j,x\rangle\\
&= \langle x,x\rangle.
\end{aligned}$$

On the other hand, x can be written as $x=\sum_{j\in J}\alpha_j e_j$. Thus, we have

$$\begin{aligned}
\langle x,x\rangle &= \Big\langle \sum_{j\in J}\alpha_j e_j,\sum_{j\in J}\alpha_j e_j\Big\rangle\\
&= \sum_{j\in J}\langle x,x_j\rangle\langle x_j,x\rangle.
\end{aligned}$$

Hence $\{x_j\}_{j\in J}$ is a Parseval or a normalized tight frame with frame bound 1.

Lemma 1.1. *([41]) Let $\{x_j\}_{j\in J}$ be a Bessel sequence of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Then the analysis operator $T\colon \mathcal{H}\to \ell^2(\mathcal{A})$ defined by*

$$Tx=\sum_{j\in J}\langle x,x_j\rangle e_j$$

is adjointable and fulfills $T^*e_j=x_j$ for all j.

We have the following equivalent definition for Bessel sequences in Hilbert C^*-modules. The main advantage of an equivalent definition of frames in Hilbert C^*-module is that it is much easier to compare the norms of two elements than to compare two elements in C^*-algebras.

Lemma 1.2. *([41]) Let $\{x_j\}_{j\in J}$ be a sequence of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Then $\{x_j\}_{j\in J}$ is a Bessel sequence with bound D if and only if*

$$\sum_{j\in J}\langle x,x_j\rangle\langle x_j,x\rangle \le D\|x\|^2$$

holds for all $x \in \mathcal{H}$.

Proposition 1.3. *([41]) Let $\mathcal{H}$ be a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$ and $\{x_j\}_{j\in J} \subseteq \mathcal{H}$ a sequence. Then $\{x_j\}_{j\in J}$ is a frame of $\mathcal{H}$ with bounds C and D if and only if*

$$C\|x\|^2 \leq \sum_{j\in J} \langle x, x_j\rangle\langle x_j, x\rangle \leq D\|x\|^2$$

for all $x \in \mathcal{H}$.

The following result characterizes Bessel sequences in terms of operators in Hilbert C^*-modules.

Proposition 1.4. *([41]) Let $\{x_j\}_{j\in J}$ be a sequence of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Then $\{x_j\}_{j\in J}$ is a Bessel sequence with Bessel bound D if and only if the operator $U\colon \ell^2(\mathcal{A}) \to \mathcal{H}$ defined by*

$$U\{c_j\}_{j\in J} = \sum_{j\in J} c_j x_j$$

is a well-defined bounded operator from $\ell^2(\mathcal{A})$ into $\mathcal{H}$ with $\|U\| \leq \sqrt{D}$.

The following result gives a necessary and sufficient perturbation theorem of frames in a Hilbert C^*-module.

Theorem 1.4. *([41]) Suppose that $\mathcal{H}$ is a Hilbert C^*-module. Let $\{x_j\}_{j\in J}$ be a frame for $\mathcal{H}$ with frame bounds C_X and D_X and $\{y_j\}_{j\in J}$ be a sequence of $\mathcal{H}$. Then the following statements are equivalent:*

1. *$\{y_j\}_{j\in J}$ is a frame of $\mathcal{H}$.*

2. *There is a constant $M > 0$ so that for all $x \in \mathcal{H}$ we have*

$$\|\sum_{j\in J} \langle x, x_j - y_j\rangle\langle x_j - y_j, x\rangle\| \leq M\|\sum_{j\in J} \langle x, x_j\rangle\langle x_j, x\rangle\|$$

and

$$\|\sum_{j\in J} \langle x, x_j - y_j\rangle\langle x_j - y_j, x\rangle\| \leq M\|\sum_{j\in J} \langle x, y_j\rangle\langle y_j, x\rangle\|.$$

Moreover, if $\{y_j\}_{j\in J}$ is a Bessel sequence, then (1) and (2) are equivalent to

3. *There exists a constant $M > 0$ so that*

$$\|\sum_{j \in J} \langle x, x_j - y_j \rangle \langle x_j - y_j, x \rangle\| \leq M \|\sum_{j \in J} \langle x, y_j \rangle \langle y_j, x \rangle\|$$

holds for all $x \in \mathcal{H}$.

CHAPTER 2

Representation of frames as regular k-distance sets

In this chapter, we introduce the concept of a regular k-distance frame in Hilbert space. Here, we discuss various characteristics of regular k-distance sets and focus on k-distance tight frames for the underlying space. We also discuss the dual frames for regular k-distance sets and provide some examples. In the end, we establish a perturbation result for regular k-distance frames.

2.1 Introduction and Preliminaries

Frames have shown to be very useful in a variety of applications. Regular two-distance is a special type of category of the frame which has some nice properties. Recently, authors make a deep study of regular two-distance sets [17].
Now we recall some basic definitions from the literature.

Definition 2.1. *A set $X \subset \mathbb{R}^n$ is called a two-distance set if there are two numbers p and q such that the distances between any pairs of points of X are either p or q.*

A two distance set X is called a spherical two-distance set if it lies in the unit sphere of $\mathbb{R}^n$. To put in another way, a set of unit vectors X in the Euclidean space $\mathbb{R}^n$ is a spherical two-distance set if there are two real numbers α and β, $-1 \leq \alpha, \beta \leq 1$ such that the inner product of any two vectors of X are either α or β. We will say that α and β are the angles of X.
For a set of vectors $X = \{x_i\}_{i=1}^{m}$ in $\mathbb{R}^n$, its *Gram matrix* G is the $m \times m$ matrix with en-

tries $G_{ij} = \langle x_i, x_j \rangle$ for $i, j \in [m]([m] = \{1, 2, 3, ..., m\}$, where m is any natural number).

$$G = \begin{bmatrix} \langle x_1, x_1 \rangle & \langle x_1, x_2 \rangle & ... & \langle x_1, x_m \rangle \\ \langle x_2, x_1 \rangle & \langle x_2, x_2 \rangle & ... & \langle x_2, x_m \rangle \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ \langle x_m, x_1 \rangle & \langle x_m, x_2 \rangle & ... & \langle x_m, x_m \rangle \end{bmatrix}.$$

Let $X = \{x_i\}_{i=1}^m$ be a spherical two-distance set in $\mathbb{R}^n$ at angle α and β. For each $i \in [m]$, we define the sets

$$I_i^\alpha = \{j \in [m] : \langle x_i, x_j \rangle = \alpha\},\ I_i^\beta = \{j \in [m] : \langle x_i, x_j \rangle = \beta\}. \tag{2.1}$$

It is easy to see that $|I_i^\alpha| + |I_i^\beta| = m - 1$ for all $i \in [m]$.

Definition 2.2. *[17] A spherical two-distance set $X = \{x_i\}_{i=1}^m$ in $\mathbb{R}^n$ at angles α and β is said to be regular if the cardinality of the set I_i^α (and hence the set I_i^β) does not depend on i. We call this number $k_\alpha = |I_i^\alpha|$, and $k_\beta = |I_i^\beta|$, the multiplicities of α and β, respectively.*

Example 2.1. *A pentagon is a regular two-distance set in $\mathbb{R}^2$.*

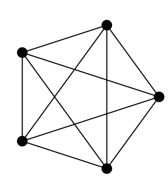

Figure 1.1

The following theorem gives a simple characterization of regular two-distance sets.

Theorem 2.1 ([17])**.** *A spherical two-distance set is regular if and only if its Gram matrix has constant row sum.*

With the help of frame potential, we deliver important characterization of tight frames.

Definition 2.3. *Let $X = \{x_i\}_{i=1}^m$ be a collection of vectors in $\mathbb{R}^n$. The frame potential for X is the quantity*

$$FP(X) = \sum_{i=1}^{m} \sum_{j=1}^{m} |\langle x_i, x_j \rangle|^2. \tag{2.2}$$

Theorem 2.2. *[7] Let $m \geq n$. If $X = \{x_i\}_{i=1}^m$ is any set of unit norm vectors in $\mathbb{R}^n$, then*

$$FP(X) \geq \frac{m^2}{n}, \tag{2.3}$$

and equality holds if and only if X is a tight frame.

Corollary 2.1. *[17] Let X be a regular two-distance set of m vectors in $\mathbb{R}^n$ at angles α, β with respective multiplicities k_α, k_β. Then*

$$1 + k_\alpha \alpha^2 + k_\beta \beta^2 \geq \frac{m}{n}, \tag{2.4}$$

and equality holds if and only if X is a two-distance tight frame.

Definition 2.4. *A set of vectors $X = \{x_i\}_{i=1}^m$ in $\mathbb{R}^n$ is said to be balanced if $\sum\limits_{i=1}^{m} x_i = 0$.*

Proposition 2.1. *[17] A set $X = \{x_i\}_{i=1}^m$ is balanced if and only if each row sum of its Gram matrix is zero.*

2.2 Main Results

In this section, we undertake a deep study of k-distance sets and we investigate the case where spherical k-distance sets form frames for the underlying spaces.

Definition 2.5. *A set X in Euclidean space $\mathbb{R}^n$ is called a k-distance set if there are k numbers a_1, a_2, ..., and a_k such that the distances between any pairs of points of X are either a_1, or a_2, ..., or a_k.*

Definition 2.6. *A set of unit vectors in n-dimensional Euclidean space is a spherical k-distance set if there are k real numbers α_1, α_2, ..., and α_k, $-1 \leq \alpha_1, \alpha_2, ..., \alpha_k \leq 1, \alpha_i \neq \alpha_j$ for $i \neq j$ such that the inner product of any two vectors of X are either α_1 or α_2, ..., or α_k. We will say that α_1, α_2, ..., and α_k are the angles of X.*

Let $X = \{x_i\}_{i=1}^m$ be a spherical k-distance set in $\mathbb{R}^n$ at angle α_1, α_2, ..., and α_k. For each $i \in [m]$, we define the sets

$$I_i^{\alpha_1} = \{j \in [m] : \langle x_i, x_j \rangle = \alpha_1\}$$
$$I_i^{\alpha_2} = \{j \in [m] : \langle x_i, x_j \rangle = \alpha_2\}$$
$$\vdots$$
$$I_i^{\alpha_k} = \{j \in [m] : \langle x_i, x_j \rangle = \alpha_k\}$$

It is clear that $|I_i^{\alpha_1}| + |I_i^{\alpha_2}| + ... + |I_i^{\alpha_k}| = m - 1$ for all $i \in [m]$.
Generally, the cardinalities of these sets, $I_i^{\alpha_1}$, $I_i^{\alpha_2}$, ..., and $I_i^{\alpha_k}$, depending on i. When they are independent with i, we say that the set is regular.

Definition 2.7. *A spherical k-distance set $X = \{x_i\}_{i=1}^m$ in $\mathbb{R}^n$ at angles α_1, α_2, ..., and α_k is said to be regular if the cardinality of the set $I_i^{\alpha_1}$ (and hence the set $I_i^{\alpha_2}$, ..., $I_i^{\alpha_k}$) does not depend on i. We call this number $k_{\alpha_1} = |I_i^{\alpha_1}|$, $k_{\alpha_2} = |I_i^{\alpha_2}|$, ..., and $k_{\alpha_k} = |I_i^{\alpha_k}|$, the multiplicities of α_1, α_2, ..., and α_k, respectively.*

For a regular k-distance set X, the sum of the entries in every row of its Gram matrix is the same. We will call this common number the Grammian constant (c) of X. This constant is always greater than or equal to zero and less than the cardinality of X. When k-distance sets form frames and tight frames for the space, we call them k-distance frames and k-distance tight frames, respectively.
Let $\{\phi_j\}_{j\in J}$ be a regular k-distance frame of a Hilbert space H and J be an index set, then we define the corresponding pre-frame operator, analysis operator, and frame operator as follows.
The operator $T : H \to \ell^2$ defined by

$$Tf = \{\langle f, \phi_j \rangle\}_{j\in J}, \ \forall \ f \in H$$

is called the *analysis operator*. The adjoint operator $T^* : \ell^2 \to H$ is given by

$$T^*\{c_j\} = \sum_{j \in J} c_j \phi_j.$$

T^* is called *pre-frame operator or the synthesis operator*. By composing T and T^*, we obtain the *frame operator* $S : H \to H$ defined by

$$Sf = T^*Tf = \sum_{j \in J} \langle f, \phi_j \rangle \phi_j, \ \forall \ f \in H.$$

We now provide an example of a regular three-distance set.

Example 2.2. *Insert a cube inside a sphere with radius 1. That is the corners of the cube must touch the surface of the sphere. There are eight corner points.*
$X = \{(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}),$
$(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}), (\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}})\}.$
Here $\alpha = \frac{1}{3}, \beta = \frac{-1}{3}$ *and* $\gamma = -1$.
The Gram matrix G is

$$G = \begin{bmatrix} 1 & 1/3 & 1/3 & 1/3 & -1/3 & -1/3 & -1/3 & -1 \\ 1/3 & 1 & -1/3 & -1/3 & 1/3 & -1 & 1/3 & -1/3 \\ 1/3 & -1/3 & 1 & -1/3 & 1/3 & 1/3 & -1 & -1/3 \\ 1/3 & -1/3 & -1/3 & 1 & -1 & 1/3 & 1/3 & -1/3 \\ -1/3 & 1/3 & 1/3 & -1 & 1 & -1/3 & -1/3 & 1/3 \\ -1/3 & -1 & 1/3 & 1/3 & -1/3 & 1 & -1/3 & 1/3 \\ -1/3 & 1/3 & -1 & 1/3 & -1/3 & -1/3 & 1 & 1/3 \\ -1 & -1/3 & -1/3 & -1/3 & 1/3 & 1/3 & 1/3 & 1 \end{bmatrix}$$

$I_1^\alpha = \{\langle x_1, x_2 \rangle, \langle x_1, x_3 \rangle, \langle x_1, x_4 \rangle\}, I_1^\beta = \{\langle x_1, x_5 \rangle, \langle x_1, x_6 \rangle, \langle x_1, x_7 \rangle\}, I_1^\gamma = \{\langle x_1, x_8 \rangle\}$
$I_2^\alpha = \{\langle x_2, x_1 \rangle, \langle x_2, x_5 \rangle, \langle x_2, x_7 \rangle\}, I_2^\beta = \{\langle x_2, x_3 \rangle, \langle x_2, x_4 \rangle, \langle x_2, x_8 \rangle\}, I_2^\gamma = \{\langle x_2, x_6 \rangle\}$
$I_3^\alpha = \{\langle x_3, x_1 \rangle, \langle x_3, x_5 \rangle, \langle x_3, x_6 \rangle\}, I_3^\beta = \{\langle x_3, x_2 \rangle, \langle x_3, x_4 \rangle, \langle x_3, x_8 \rangle\}, I_3^\gamma = \{\langle x_3, x_7 \rangle\}$
$I_4^\alpha = \{\langle x_4, x_1 \rangle, \langle x_4, x_6 \rangle, \langle x_4, x_7 \rangle\}, I_4^\beta = \{\langle x_4, x_2 \rangle, \langle x_4, x_3 \rangle, \langle x_4, x_8 \rangle\}, I_4^\gamma = \{\langle x_4, x_5 \rangle\}$

$I_5^\alpha = \{\langle x_5, x_2\rangle, \langle x_5, x_3\rangle, \langle x_5, x_8\rangle\}, I_5^\beta = \{\langle x_5, x_1\rangle, \langle x_5, x_6\rangle, \langle x_5, x_7\rangle\}, I_5^\gamma = \{\langle x_5, x_4\rangle\}$

$I_6^\alpha = \{\langle x_6, x_3\rangle, \langle x_6, x_4\rangle, \langle x_6, x_8\rangle\}, I_6^\beta = \{\langle x_6, x_1\rangle, \langle x_6, x_5\rangle, \langle x_6, x_7\rangle\}, I_6^\gamma = \{\langle x_6, x_2\rangle\}$

$I_7^\alpha = \{\langle x_7, x_2\rangle, \langle x_7, x_4\rangle, \langle x_7, x_8\rangle\}, I_7^\beta = \{\langle x_7, x_1\rangle, \langle x_7, x_5\rangle, \langle x_6, x_1\rangle\}, I_7^\gamma = \{\langle x_7, x_3\rangle\}$

$I_8^\alpha = \{\langle x_8, x_5\rangle, \langle x_8, x_6\rangle, \langle x_8, x_7\rangle\}, I_6^\beta = \{\langle x_8, x_2\rangle, \langle x_8, x_3\rangle, \langle x_8, x_4\rangle\}, I_8^\gamma = \{\langle x_8, x_1\rangle\}$

Here $c = 0$ and $k_\alpha = 3, k_\beta = 3, k_\gamma = 1$.

We now introduce the definition of dual frames for regular k-distance set. To reconstruct a vector from its frame coefficients, we require the notion of dual frame.

Definition 2.8. *Let $\{\phi_j\}_{j\in J}$ be a regular k-distance frame. Then there is another regular k-distance frame $\{\tilde{\phi}_j\}_{j\in J} \subset H$, such that*

$$\{\tilde{\phi}_j\} = (T^*T)^{-1}\phi_j = S^{-1}\phi_j, \;\; j \in J. \tag{2.5}$$

The family $\{S^{-1}\phi_j\}_{j\in J}$ is also a regular k-distance frame for H, called the *canonical dual frame*.

In general, a regular k-distance frame $\{\psi_j\}_{j\in J} \subset H$ is called *alternate dual or simply a dual* for $\{\phi_j\}_{j\in J}$ if

$$f = \sum_{j\in J}\langle f, \psi_j\rangle\phi_j, \; \forall\, f \in H. \tag{2.6}$$

Theorem 2.3. *Suppose $\{\phi_j\}_{j\in J}$ is a regular k-distance frame of a Hilbert space H, with associated frame operator $S = T^*T$ and frame bounds $0 < A \leq B < \infty$. Then the set $\{\tilde{\phi}_j = (T^*T)^{-1}\phi_j = S^{-1}\phi_j\}$ is another regular k-distance frame of H, with*

$$\frac{1}{B}\|f\|^2 \leq \|\tilde{T}f\|^2 \leq \frac{1}{A}\|f\|^2, \; \forall\, f \in H.$$

The set $\{\tilde{\phi}_j\}$ is called the canonical dual frame associated with the original regular k-distance frame.

Proof. If $\{\tilde{\phi}_j\}$ is to be a regular k-distance frame then there is some operator $\tilde{T}$

satisfying

$$\tilde{T}f = \langle f, \tilde{\phi}_j \rangle = \langle f, (T^*T)^{-1}\phi_j \rangle.$$

$(T^*T)^{-1}$ is the inverse of a bounded self-adjoint operator, so it is also self-adjoint, and

$$(\tilde{T}f)_j = \langle f, (T^*T)^{-1}\phi_j \rangle.$$

By definition this is the j^{th} component of $T(T^*T)^{-1}$. Thus,

$$\tilde{T} = T(T^*T)^{-1}.$$

We also have

$$\tilde{T}^* = (T^*T)^{-1}T^*.$$

and

$$\begin{aligned} \|\tilde{T}f\|^2 &= \langle \tilde{T}^*\tilde{T}f, f \rangle \\ &= \langle (T^*T)^{-1}T^*T(T^*T)^{-1}f, f \rangle \\ &= \langle (T^*T)^{-1}f, f \rangle. \end{aligned}$$

Let $g = (T^*T)^{-1}f$, so that

$$\begin{aligned} \|\tilde{T}f\|^2 &= \langle (T^*T)^{-1}f, f \rangle \\ &= \langle g, T^*Tg \rangle \\ &= \langle Tg, Tg \rangle \\ &= \|Tg\|^2. \end{aligned}$$

Since $\{\phi_j\}_{j\in J}$ is a regular k-distance frame for H, we have

$$A\|g\|^2 \leq \|Tg\|^2 \leq B\|g\|^2, \; g \in H.$$

After rearranging the above inequality, we get

$$\frac{1}{B}\|Tg\|^2 \leq \|g\|^2 \leq \frac{1}{A}\|Tg\|^2.$$

Inserting $g = (T^*T)^{-1}f$ back into this inequality, we have

$$\frac{1}{B}\|\tilde{T}f\|^2 \leq \|(T^*T)^{-1}f\|^2 \leq \frac{1}{A}\|\tilde{T}f\|^2 \quad (\because \|\tilde{T}f\|^2 = \|Tg\|^2).$$

This implies

$$\begin{aligned}\frac{1}{B}\langle \tilde{T}f, \tilde{T}f\rangle &\leq \|S^{-1}f\|^2 \leq \frac{1}{A}\langle \tilde{T}f, \tilde{T}f\rangle \\ \frac{1}{B}\langle \tilde{T}^*\tilde{T}f, f\rangle &\leq \langle (S^{-1})^2 f, f\rangle \leq \frac{1}{A}\langle \tilde{T}^*\tilde{T}f, f\rangle.\end{aligned}$$

This gives us

$$\frac{1}{B}\langle f, f\rangle \leq \langle S^{-1}f, f\rangle \leq \frac{1}{A}\langle f, f\rangle.$$

If $\{\phi_j\}_{j\in J}$ is a regular k-distance set with distances $\alpha_1, \alpha_2, ...,$ and α_k then by direct computation, one can see that $\{\tilde{\phi}_j\}$ is a regular k-distance set with distances $\|S^{-1}\|\alpha_1, \|S^{-1}\|\alpha_2, ...,$ and $\|S^{-1}\|\alpha_k$. Also, $\{\tilde{\phi}_j\}$ is a regular k-distance frame with bound $0 < \frac{1}{B} \leq \frac{1}{A} \leq \infty$ and $S^{-1} = \tilde{S}$ is the frame operator for a regular k-distance frame $\{\tilde{\phi}_j\}$. □

Example 2.3. *The set* $X = \{(0,1), (0,-1), (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})\}$ *is a regular three-distance frame for* $\mathbb{R}^2$.

The Gram matrix is given by

$$G = \begin{bmatrix} 1 & -1 & 1/\sqrt{2} & -1/\sqrt{2} \\ -1 & 1 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 1 & -1 \\ -1/\sqrt{2} & 1/\sqrt{2} & -1 & 1 \end{bmatrix}.$$

Here $\alpha = \frac{1}{\sqrt{2}}, \beta = \frac{-1}{\sqrt{2}}$ *and* $\gamma = -1$.

The analysis operator is

$$T = \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

and the synthesis operator is given by

$$T^* = \begin{bmatrix} 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1 & -1 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

The frame operator $S = T^*T$

$$S = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}.$$

Here X *is a regular three-distance frame with lower and upper bounds* $A = 0.58$ *and* $B = 3.41$*, respectively.*

Now

$$S^{-1} = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

and

$$\tilde{T} = TS^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & -1/2 \\ -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \end{bmatrix}.$$

Therefore, the set $Y = \left\{ (\frac{1}{2}, \frac{1}{2}), (\frac{-1}{2}, \frac{-1}{2}), (\frac{-1}{\sqrt{2}}, 0), (\frac{1}{\sqrt{2}}, 0) \right\}$ *is a canonical dual frame of* X *with lower and upper frame bounds* 0.3 *and* 1.7*, respectively. Also, the set* Y *is a regular three-distance set if we convert the set of vectors in* Y *to unit vectors as* $\left\{ (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}), (-1, 0), (1, 0) \right\}$*. The Gram matrix for the canonical dual frame* *of X is as follows:*

$$\tilde{G} = \begin{bmatrix} 1 & -1 & -1/\sqrt{2} & 1/\sqrt{2} \\ -1 & 1 & 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} & 1 & -1 \\ 1/\sqrt{2} & -1/\sqrt{2} & -1 & 1 \end{bmatrix}.$$

Proposition 2.2. *Let $X = \{x_i\}_{i=1}^{m}$ be a regular k-distance set of m vectors in $\mathbb{R}^n$ at angles $\alpha_1, \alpha_2, ...,$ and α_k with respective multiplicities $k_{\alpha_1}, k_{\alpha_2}, ...,$ and k_{α_k}. Then*

$$1 + k_{\alpha_1}\alpha_1^2 + k_{\alpha_2}\alpha_2^2 + ... + k_{\alpha_k}\alpha_k^2 \geq \frac{m}{n} \tag{2.7}$$

and equality holds if and only if X is a k-distance tight frame.

Proof. We know that

$$FP(X) \geq \frac{m^2}{n}. \tag{2.8}$$

Also,

$$FP(X) = m[k_{\alpha_1}\alpha_1^2 + k_{\alpha_2}\alpha_2^2 + ... + k_{\alpha_k}\alpha_k^2 + 1]. \tag{2.9}$$

Using equation (2.8) and (2.9), we get

$$\begin{aligned} m[k_{\alpha_1}\alpha_1^2 + k_{\alpha_2}\alpha_2^2 + ... + k_{\alpha_k}\alpha_k^2 + 1] &\geq \frac{m^2}{n} \\ [k_{\alpha_1}\alpha_1^2 + k_{\alpha_2}\alpha_2^2 + ... + k_{\alpha_k}\alpha_k^2 + 1] &\geq \frac{m}{n}. \end{aligned}$$

The equality part of the theorem holds true as we know that $FP(X) = \dfrac{m^2}{n}$ if and only if X is a tight frame. □

Tight frames are those frames in which the frame bounds are equal. For tight frames, we do not require the inverse of a frame operator. They play a key role in wide applications as tight frames look-like a more natural way to reconstruct vectors. Tight frames are closest to orthonormal bases as they are redundant set of vectors and have properties like bases.

Theorem 2.4. *Assume $X = \{x_i\}_{i=1}^m$ is a regular k-distance frame in $\mathbb{R}^n$ with angles α_1, α_2, ..., and α_k. Then the following are equivalent:*
(1) X is a m/n-tight frame.
(2) For some $J \subset [m]$ with span $\{x_i\}_{i \in J} = \mathbb{R}^n$,

$$\alpha_1 \sum_{j \in I_i^{\alpha_1}} x_j + \alpha_2 \sum_{j \in I_i^{\alpha_2}} x_j + ... + \alpha_k \sum_{j \in I_i^{\alpha_k}} x_j = \big(\frac{m}{n} - 1\big) x_i, \textit{ for every } i \in J.$$

Proof. First we assume that $X = \{x_i\}_{i=1}^m$ is a regular k-distance m/n-tight frame. So, for any $i \in [m]$, we deduce

$$\begin{aligned} \frac{m}{n} x_i &= \sum_{j=1}^m \langle x_i, x_j \rangle x_j \\ &= \sum_{j \in I_i^{\alpha_1}} \langle x_i, x_j \rangle x_j + \sum_{j \in I_i^{\alpha_2}} \langle x_i, x_j \rangle x_j + ... + \sum_{j \in I_i^{\alpha_k}} \langle x_i, x_j \rangle x_j + \langle x_i, x_i \rangle x_i \\ &= \alpha_1 \sum_{j \in I_i^{\alpha_1}} x_j + \alpha_2 \sum_{j \in I_i^{\alpha_2}} x_j + ... + \alpha_k \sum_{j \in I_i^{\alpha_k}} x_j + x_i. \end{aligned}$$

This implies

$$\alpha_1 \sum_{j \in I_i^{\alpha_1}} x_j + \alpha_2 \sum_{j \in I_i^{\alpha_2}} x_j + ... + \alpha_k \sum_{j \in I_i^{\alpha_k}} x_j = \big(\frac{m}{n} - 1\big) x_i, \text{ for all } i \in J.$$

for some $J \subset [m]$ with span $\{x_i\}_{i \in J} = \mathbb{R}^n$.
For the converse part, we have

$$\alpha_1 \sum_{j \in I_i^{\alpha_1}} x_j + \alpha_2 \sum_{j \in I_i^{\alpha_2}} x_j + ... + \alpha_k \sum_{j \in I_i^{\alpha_k}} x_j = \big(\frac{m}{n} - 1\big) x_i, \text{ for every } i \in J.$$

From this it follows that

$$\begin{aligned} \frac{m}{n} x_i &= \sum_{j=1}^m \langle x_i, x_j \rangle x_j \\ \Longrightarrow \frac{m}{n} x_i &= S x_i, \text{ for every } i \in J, \end{aligned}$$

where S is the frame operator of $X = \{x_i\}_{i=1}^m$. Now $Sx = \dfrac{m}{n}x$, for every $x \in \mathbb{R}^n$ follows from $\mathrm{span}\{x_i\}_{i\in J} = \mathbb{R}^n$. Therefore, $X = \{x_i\}_{i=1}^m$ is a m/n-tight frame. $\square$

The following theorem shows an approach to creating balanced, regular k-distance sets in one lower dimension using a non-balanced set.

Theorem 2.5. *Let $X = \{x_i\}_{i=1}^m$ be a regular k-distance set of distinct vectors in $\mathbb{R}^n$ with its Grammian constant c. Let α_1, α_2, ..., and α_k be its angles with multiplicities k_{α_1}, k_{α_2}, ..., and k_{α_k}, respectively. Assume that X is not balanced and let P be the orthogonal projection onto span$\{z\}$, where $z = \sum_{i=1}^m x_i$. Then $Y = \{\frac{(I-P)x_i}{\|(I-P)x_i\|}\}_{i=1}^m$ is a balanced, regular k-distance set of distinct vectors in $\mathbb{R}^{n-1}$ at angles $\frac{m}{m-c}(\alpha_1 - \frac{c}{m})$, $\frac{m}{m-c}(\alpha_2 - \frac{c}{m})$, ..., and $\frac{m}{m-c}(\alpha_k - \frac{c}{m})$ with respective multiplicities k_{α_1}, k_{α_2}, ..., and k_{α_k}.*

Proof. For every $x \in \mathbb{R}^n$, we have that

$$Px = \left\langle x, \frac{z}{\|z\|} \right\rangle \frac{z}{\|z\|}.$$

Now, we compute

$$\|z\|^2 = \langle \sum_{j=1}^m x_i, \sum_{j=1}^m x_j \rangle = mc,$$

and for all i,

$$\begin{aligned}
\|(I-P)x_i\|^2 &= \|x_i\|^2 - \|Px_i\|^2 \\
&= \langle x_i, x_i \rangle - \langle Px_i, Px_i \rangle \\
&= 1 - \frac{1}{\|z\|^2} |\langle x_i, z \rangle|^2 \quad (\because X \text{ is a regular distance set}) \\
&= 1 - \frac{c^2}{mc} \\
&= \frac{m-c}{m}.
\end{aligned}$$

If we set $y_i = \dfrac{(I-P)x_i}{\|(I-P)x_i\|}$, then

$$\begin{aligned}
\langle y_i, y_j\rangle &= \frac{m}{m-c}(\langle x_i, x_j\rangle - \langle Px_i, Px_j\rangle) \\
&= \frac{m}{m-c}(\langle x_i, x_j\rangle - \frac{1}{\|z\|^2}\langle x_i, z\rangle\langle x_j, z\rangle) \\
&= \frac{m}{m-c}(\langle x_i, x_j\rangle - \frac{c^2}{\|z\|^2}) \\
&= \frac{m}{m-c}(\langle x_i, x_j\rangle - \frac{c}{m}). \qquad (2.10)
\end{aligned}$$

This shows that Y is also regular k-distance set with the same multiplicities as of X because X is regular k-distance set. One can easily see that the vectors in the set Y are distinct since

$$\frac{m}{m-c}(\langle x_i, x_j\rangle - \frac{c}{m}) = 1 \text{ if and only if } \langle x_i, x_j\rangle = 1.$$

Now, we compute the Grammian constant for the set Y. So, for any i (let's say $i = 1$), we have

$$\begin{aligned}
&\sum_{j=1}^{m}\langle y_1, y_j\rangle \\
&= \langle y_1, y_1\rangle + \langle y_1, y_2\rangle + \cdots + \langle y_1, y_m\rangle \\
&= \frac{m}{m-c}\Big(\langle x_1, x_1\rangle - \frac{c}{m}\Big) + \frac{m}{m-c}\Big(\langle x_1, x_2\rangle - \frac{c}{m}\Big) + \cdots + \frac{m}{m-c}\Big(\langle x_1, x_m\rangle - \frac{c}{m}\Big) \quad \text{(Using (2.10))} \\
&= \frac{m}{m-c}\Big[\Big(\langle x_1, x_1\rangle - \frac{c}{m}\Big) + \Big(\langle x_1, x_2\rangle - \frac{c}{m}\Big) + \cdots + \Big(\langle x_1, x_m\rangle - \frac{c}{m}\Big)\Big] \\
&= \frac{m}{m-c}\Big[\langle x_1, x_1\rangle + \langle x_1, x_2\rangle + \cdots + \langle x_1, x_m\rangle - m\Big(\frac{c}{m}\Big)\Big] \\
&= 0 \ \big(\because \ \sum_{j=1}^{m}\langle x_1, x_j\rangle = c\big).
\end{aligned}$$

This implies that Y is balanced set. □

Proposition 2.3. *Let $X = \{x_i\}_{i=1}^m$ be a k-distance tight frame for $\mathbb{R}^n$ at angles α_1, α_2, ..., and α_k. Also, $\alpha_i \neq -\alpha_j$ for $i \neq j$ and $|I_i^{\alpha_2}|, \ldots, |I_i^{\alpha_k}|$ independent of i, then X is regular. In addition to that the Grammian constant of X is either* 0 *or* $\dfrac{m}{n}$.

Proof. By using second part of Theorem 2.4 , we have

$$\alpha_1 \sum_{j\in I_i^{\alpha_1}} x_j + \alpha_2 \sum_{j\in I_i^{\alpha_2}} x_j + ... + \alpha_k \sum_{j\in I_i^{\alpha_k}} x_j = (\frac{m}{n} - 1)x_i, \quad \forall\, i \in [m].$$

Now, by taking the inner product on both sides of above equation with x_i, we get

$$|I_i^{\alpha_1}|\alpha_1^2 + |I_i^{\alpha_2}|\alpha_2^2 + \cdots + |I_i^{\alpha_k}|\alpha_k^2 = \frac{m}{n} - 1.$$

Using $|I_i^{\alpha_1}| + |I_i^{\alpha_2}| + \cdots + |I_i^{\alpha_k}| = m - 1$ in the above equation, we have

$$|I_i^{\alpha_1}|\alpha_1^2 + |I_i^{\alpha_2}|\alpha_2^2 + \cdots + |I_i^{\alpha_{k-1}}|\alpha_{k-1}^2 + (m - 1 - |I_i^{\alpha_1}| - |I_i^{\alpha_2}| - \cdots - |I_i^{\alpha_{k-1}}|)\alpha_k^2 = \frac{m}{n} - 1$$
$$\implies |I_i^{\alpha_1}|(\alpha_1^2 - \alpha_k^2) + |I_i^{\alpha_2}|(\alpha_2^2 - \alpha_k^2) + \cdots + |I_i^{\alpha_{k-1}}|(\alpha_{k-1}^2 - \alpha_k^2) + (m-1)\alpha_k^2 = \frac{m}{n} - 1$$

Solving for $|I_i^{\alpha_1}|$, we get

$$|I_i^{\alpha_1}| = \frac{\left(\frac{m}{n} - 1\right) + (1-m)\alpha_k^2 + |I_i^{\alpha_2}|(\alpha_k^2 - \alpha_2^2) + ... + |I_i^{\alpha_{k-1}}|(\alpha_k^2 - \alpha_{k-1}^2)}{\alpha_1^2 - \alpha_k^2}$$

which is independent of i, if $\alpha_i \neq -\alpha_j$ for $i \neq j$ and $|I_i^{\alpha_2}|, ..., |I_i^{\alpha_k}|$ independent of i. Therefore X is regular.

Now we want to prove that the Grammian constant of X is either 0 or $\frac{m}{n}$. Since X is a finite unit-norm tight frame for $\mathbb{R}^n$, we can say that Gram matrix G of X has one nonzero eigenvalue $\frac{m}{n}$ of multiplicity n and an eigenvalue 0 of multiplicity $m - n$ i.e. G has only two eigenvalue $\frac{m}{n}$ and 0.

For all $i \in [m]$

$$\sum_{j=1}^{m} \langle x_i, x_j \rangle = \sum_{j=1}^{m} G_{ij} = c.$$

Then we have

$$G\mathbb{1} = c\mathbb{1},$$

where $\mathbb{1}$ denote the vector of all 1's. Also we observe that the row sum of the Gram matrix G is an eigenvalue of G. This implies that the Grammian constant (c) of X is either 0 or $\frac{m}{n}$. □

Now the next result shows that there is a constraint on the cardinalities of the set $I_i^{\alpha_1}, I_i^{\alpha_2}, \cdots$, and $I_i^{\alpha_k}$ if a regular k-distance set in $\mathbb{R}^n$ contains odd number of elements.

Proposition 2.4. *Let $X = \{x_i\}_{i=1}^m$ be a regular k-distance set in $\mathbb{R}^n$ at angles $\alpha_1, \alpha_2, ...,$ and α_k with multiplicities $k_{\alpha_1}, k_{\alpha_2}, ...,$ and k_{α_k}, respectively. Then $k_{\alpha_1}, k_{\alpha_2}, ...,$ and k_{α_k} are even if m is odd.*

Proof. The Gram matrix G of X is defined by $G_{ij} = \langle x_i, x_j \rangle, 1 \leq i, j \leq m$ which is a $m \times m$ self-adjoint (i.e. $G = G^*$) matrix. Now as we have X is a regular k-distance set of m vectors in $\mathbb{R}^n$, each row of G has exactly k_{α_1} elements α_1, k_{α_2} elements α_2, ..., and k_{α_k} elements α_k. Also, we have

$$k_{\alpha_1} + k_{\alpha_2} + \cdots + k_{\alpha_k} = m - 1 \implies mk_{\alpha_1} + mk_{\alpha_2} + \cdots + mk_{\alpha_k} = m(m-1).$$

If m is odd then $m(m-1)$ is an even number. This implies that $mk_{\alpha_1}, mk_{\alpha_2}, ...,$ and mk_{α_k} are even otherwise the Gram matrix of X would not be self-adjoint matrix and hence we deduce that $mk_{\alpha_1}, mk_{\alpha_2}, ...,$ and mk_{α_k} should be even and thus $k_{\alpha_1}, k_{\alpha_2}$, ..., and k_{α_k} are even. □

Lemma 2.1. *Let $X = \{x_i\}_{i=1}^m$ be a regular k-distance tight frame for $\mathbb{R}^n$ at angles $\alpha_1, \alpha_2, ...,$ and α_k with multiplicities $k_{\alpha_1}, k_{\alpha_2}, ...,$ and k_{α_k} respectively. Then we have*

$$1 + k_{\alpha_1}\alpha_1 + k_{\alpha_2}\alpha_2 + \cdots + k_{\alpha_k}\alpha_k = 0, \text{ or } 1 + k_{\alpha_1}\alpha_1 + k_{\alpha_2}\alpha_2 + \cdots + k_{\alpha_k}\alpha_k = \frac{m}{n},$$

and

$$1 + k_{\alpha_1}\alpha_1^2 + k_{\alpha_2}\alpha_2^2 + \cdots + k_{\alpha_k}\alpha_k^2 = \frac{m}{n}.$$

Proof. One can refer ([17]). □

Theorem 2.6. *Let $X = \{x_i\}_{i=1}^m$ be a regular k-distance tight frame in $\mathbb{R}^n$ at angles $\alpha_1, \alpha_2, ...,$ and α_k with multiplicities $k_{\alpha_1}, k_{\alpha_2}, ...,$ and k_{α_k} respectively. Let G be the Gram matrix of X with its Grammian constant c. Define a new matrix G' as*

$$\begin{aligned} G'_{ij} &= \beta - G_{ij}, \text{ for } i \neq j \\ &= 1, \text{ for } i = j \end{aligned}$$

where $\beta = \dfrac{-2}{m-1}$ *if* $c = 0$*, and* $\beta = \dfrac{2(m-n)}{n(m-1)}$ *if* $c = m/n$.
Then $G^{'}$ *has the following characteristics:*
(1) $G^{'}$ *is a* $m \times m$ *self-adjoint matrix and each row has exactly* k_{α_1} *elements* $\beta - \alpha_1$, k_{α_2} *elements* $\beta - \alpha_2$, ..., *and* k_{α_k} *elements* $\beta - \alpha_k$.
(2) $G^{'}$ *has constant row sum. Indeed,*

$$1 + k_{\alpha_1}(\beta - \alpha_1) + k_{\alpha_2}(\beta - \alpha_2) + \cdots + k_{\alpha_k}(\beta - \alpha_k) = 0 \text{ if } \beta = \frac{-2}{m-1}, \text{ and}$$
$$1 + k_{\alpha_1}(\beta - \alpha_1) + k_{\alpha_2}(\beta - \alpha_2) + \cdots + k_{\alpha_k}(\beta - \alpha_k) = m/n \text{ if } \beta = \frac{2(m-n)}{n(m-1)}.$$

(3) It also possess $1 + k_{\alpha_1}(\beta - \alpha_1)^2 + k_{\alpha_2}(\beta - \alpha_2)^2 + \cdots + k_{\alpha_k}(\beta - \alpha_k)^2 = m/n$ *for both the values of* β.

Proof. As we know G is a $m \times m$ self-adjoint matrix and each row has exactly k_{α_1} elements α_1, k_{α_2} elements α_2, ..., and k_{α_k} elements α_k. So we can deduce (1) easily.
For (2), we first consider the case $\beta = \dfrac{-2}{m-1}$.
By definition $\sum_{j=1}^{m} \langle x_i, x_j \rangle = c = 1 + k_{\alpha_1}\alpha_1 + k_{\alpha_2}\alpha_2 + \cdots + k_{\alpha_k}\alpha_k = 0$.
Thus, we have

$$\begin{aligned}
1 + k_{\alpha_1}(\beta - \alpha_1) + k_{\alpha_2}(\beta - \alpha_2) + ... + k_{\alpha_k}(\beta - \alpha_k) & \\
&= 1 - (k_{\alpha_1}\alpha_1 + k_{\alpha_2}\alpha_2 + ... + k_{\alpha_k}\alpha_k) + \beta(k_{\alpha_1} + k_{\alpha_2} + ... + k_{\alpha_k}) \\
&= 1 - (-1) + \beta(m-1) \\
&= 2 - (m-1)\frac{2}{m-1} \\
&= 0.
\end{aligned}$$

Now we consider the case for $\beta = \dfrac{2(m-n)}{n(m-1)}$.
Again by definition, we have

$$\sum_{j=1}^{m} \langle x_i, x_j \rangle = c = 1 + k_{\alpha_1}\alpha_1 + k_{\alpha_2}\alpha_2 + \cdots + k_{\alpha_k}\alpha_k = \frac{m}{n}.$$

So, we have

$$\begin{aligned}
&1 + k_{\alpha_1}(\beta - \alpha_1) + k_{\alpha_2}(\beta - \alpha_2) + ... + k_{\alpha_k}(\beta - \alpha_k) \\
&\qquad = 1 - (k_{\alpha_1}\alpha_1 + k_{\alpha_2}\alpha_2 + ... + k_{\alpha_k}\alpha_k) + \beta(k_{\alpha_1} + k_{\alpha_2} + ... + k_{\alpha_k}) \\
&\qquad = 1 - \big(\frac{m}{n} - 1\big) + \beta(m-1) \\
&\qquad = 1 - \big(\frac{m}{n} - 1\big) + (m-1)\Big[\frac{2(m-n)}{n(m-1)}\Big] \\
&\qquad = 2 - \frac{m}{n} + \frac{2(m-n)}{n} \\
&\qquad = \frac{2n - m + 2m - 2n}{n} \\
&\qquad = \frac{m}{n}
\end{aligned}$$

which is the required claim.
For (3), first we have $\beta = \dfrac{-2}{m-1}$ and $c = 1 + k_{\alpha_1}\alpha_1 + k_{\alpha_2}\alpha_2 + \cdots + k_{\alpha_k}\alpha_k = 0$
Thus,

$$\begin{aligned}
&1 + k_{\alpha_1}(\beta - \alpha_1)^2 + k_{\alpha_2}(\beta - \alpha_2)^2 + \cdots + k_{\alpha_k}(\beta - \alpha_k)^2 \\
&\quad = 1 + (k_{\alpha_1}\alpha_1^2 + k_{\alpha_2}\alpha_2^2 + \cdots + k_{\alpha_k}\alpha_k^2) - 2\beta(k_{\alpha_1}\alpha_1 + k_{\alpha_2}\alpha_2 + \cdots + k_{\alpha_k}\alpha_k) + \beta^2(m-1) \\
&\quad = 1 + \big(\frac{m}{n} - 1\big) - 2\beta(-1) + \beta^2(m-1) \quad (\because 1 + k_{\alpha_1}\alpha_1^2 + k_{\alpha_2}\alpha_2^2 + \cdots + k_{\alpha_k}\alpha_k^2 = \frac{m}{n}) \\
&\quad = \frac{m}{n} + 2\beta + \beta^2(m-1) \\
&\quad = \frac{m}{n} + 2\Big(\frac{-2}{m-1}\Big) + (m-1)\Big(\frac{-2}{m-1}\Big)^2 \\
&\quad = \frac{m}{n}.
\end{aligned}$$

Similarly, for the case $\beta = \dfrac{2(m-n)}{n(m-1)}$ and $c = 1 + k_{\alpha_1}\alpha_1 + k_{\alpha_2}\alpha_2 + \cdots + k_{\alpha_k}\alpha_k = \dfrac{m}{n}$.

We have,

$$
\begin{aligned}
&1+k_{\alpha_1}(\beta-\alpha_1)^2+k_{\alpha_2}(\beta-\alpha_2)^2+\cdots+k_{\alpha_k}(\beta-\alpha_k)^2\\
&\quad=1+(k_{\alpha_1}\alpha_1^2+k_{\alpha_2}\alpha_2^2+\cdots+k_{\alpha_k}\alpha_k^2)-2\beta(k_{\alpha_1}\alpha_1+k_{\alpha_2}\alpha_2+\cdots+k_{\alpha_k}\alpha_k)+\beta^2(m-1)\\
&\quad=1+(\frac{m}{n}-1)-2\beta(\frac{m}{n}-1)+\beta^2(m-1)\quad(\because 1+k_{\alpha_1}\alpha_1^2+k_{\alpha_2}\alpha_2^2+\cdots+k_{\alpha_k}\alpha_k^2=\frac{m}{n})\\
&\quad=\frac{m}{n}-2\beta\Big(\frac{m-n}{n}\Big)+\beta^2(m-1)\\
&\quad=\frac{m}{n}-2\Big(\frac{2(m-n)}{n(m-1)}\Big)\Big(\frac{m-n}{n}\Big)+(m-1)\Big(\frac{2(m-n)}{n(m-1)}\Big)^2\\
&\quad=\frac{m}{n}
\end{aligned}
$$

which is the required claim. □

Lemma 2.2. *Let $X=\{x_i\}_{i=1}^m$ be a regular k-distance tight frame for $\mathbb{R}^n$. Then G is the Gram matrix of X if and only if it satisfies the following conditions:*
(1) $G^2=\frac{m}{n}G$.
(2) $G_{ii}=1$ for all i.
(3) There exists α_1,α_2, ..., and α_k such that G_{ij} equals either α_1,α_2, ..., or α_k, where $\alpha_i\neq\alpha_j$ for $i\neq j$.

Proof. First we assume that G is the Gram matrix of a regular k-distance tight frame $X=\{x_i\}_{i=1}^m$ for $\mathbb{R}^n$. As we know that X is a finite unit-norm tight frame for $\mathbb{R}^n$, we can say that Gram matrix G of X has only two eigenvalue $\frac{m}{n}$ and 0. Therefore $G^2=\frac{m}{n}G$ holds true. Since all the vectors of X are unit norm thus for all i, $G_{ii}=1$. Condition (3) also holds true because X is a regular k-distance set for $\mathbb{R}^n$.
For the converse part, we assume that G satisfies conditions (1), (2) and (3) and we want to prove that G is the Gram matrix of X.
From condition (1), we have $G^2=\frac{m}{n}G$. This implies G is positive semi-definite and hence for some set of vectors G is the Gram matrix as G has only two eigenvalue $\frac{m}{n}$ and 0.
Let $\frac{m}{n}$ be an eigenvalue of G of multiplicity k as $tr(G)=m>0$. Then we have,

$$tr(G)=m=k\frac{m}{n}.$$

The above equation is only true when $k = n$ and so this implies $\text{span}\{x_j\}_{j=1}^m = \mathbb{R}^n$, where $\{x_j\}_{j=1}^m$ is set of vectors in $\mathbb{R}^n$. This shows $\{x_j\}_{j=1}^m$ is a frame for $\mathbb{R}^n$.
Now we present a method to select the vectors from $\mathbb{R}^n$.
Let D be defined as

$$D = \begin{bmatrix} \frac{m}{n}\mathbb{I} & 0 \\ 0 & 0 \end{bmatrix}$$

which is a diagonal matrix of order m and I is the identity matrix of order n.
Then there exists an unitary matrix U of eigenvectors of G such that

$$\begin{aligned} G &= UDU^* \\ &= \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \frac{m}{n}\mathbb{I} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} = \frac{m}{n} U_1 U_1^* \end{aligned}$$

where U_1 and U_2 are $m \times n$ and $m \times (m-n)$ submatrices of U whose columns are eigenvectors of G with eigenvalues $\frac{m}{n}$ and 0, respectively.
Now we choose the set of vectors to be the column of the $n \times n$ matrix $X = \sqrt{\frac{m}{n}} U_1^*$.
Since $XX^* = \frac{m}{n} U_1^* U_1 = \frac{m}{n}\mathbb{I}$, these set of vectors form a tight frame. And its Gram matrix G satisfies condition (2) which implies that these set of vectors are unit norm and (3) shows that it is a k-distance set. Thus, G is the Gram matrix of X. $\square$

Theorem 2.7. *Let X be a regular k-distance tight frame of m vectors in $\mathbb{R}^n$ at angles $\alpha_1, \alpha_2, ...,$ and α_k with multiplicities $k_{\alpha_1}, k_{\alpha_2}, ...,$ and k_{α_k} respectively. Let G be its Gram matrix and c be its Grammian constant. Let G' be defined by*

$$G' = (2 - \gamma)\mathbb{I} + \gamma\mathbb{J} - G,$$

where $\gamma = \frac{-2}{m-1}$ if $c = 0$, and $\gamma = \frac{2(m-n)}{n(m-1)}$ if $c = m/n$. Also, here $\mathbb{I}$ the identity matrix, and $\mathbb{J}$ is the matrix whose all entries are 1.
Then we have the following:
(1) For $\gamma = \frac{-2}{m-1}$, G' is the Gram matrix of a regular k-distance tight frame Y for $\mathbb{R}^n$ if and only if $m = 2n+1$.
(2) For $\gamma = \frac{2(m-n)}{n(m-1)}$, G' is the Gram matrix of a regular k-distance tight frame Y for $\mathbb{R}^n$ if and only if $m = 2n-1$.

In addition to that, the angles of Y are $\gamma - \alpha_1$ and $\gamma - \alpha_2$, ..., and $\gamma - \alpha_k$ with the same multiplicities of X, i.e., k_{α_1}, k_{α_2}, ..., k_{α_k}, respectively. Also, Y is a balanced set if $\gamma = \dfrac{-2}{m-1}$.

Proof. First, we will prove the result (1). By Lemma 2.2 and Theorem 2.6, it is sufficient to find conditions for which $G'^2 = \frac{m}{n}G'$. Also, we observe that $G\mathbb{J} = \mathbb{J}G = (1 + \alpha_1 k_{\alpha_1} + \alpha_2 k_{\alpha_2} + \cdots + \alpha_k k_{\alpha_k})\mathbb{J} = 0$, $G^2 = \frac{m}{n}G$ and $\mathbb{J}^2 = m\mathbb{J}$.
Now, we deduce that

$$\begin{aligned}
G'^2 &= [(2-\gamma)\mathbb{I} + \gamma\mathbb{J} - G]^2 \\
&= (2-\gamma)^2\mathbb{I}^2 + \gamma^2\mathbb{J}^2 + G^2 + 2\gamma(2-\gamma)\mathbb{J} - 2(2-\gamma)G - 2\gamma\mathbb{J}G \\
&= (2-\gamma)^2\mathbb{I} + \gamma^2 m\mathbb{J} + \frac{m}{n}G + 2\gamma(2-\gamma)\mathbb{J} - 2(2-\gamma)G \\
&= (2-\gamma)^2\mathbb{I} + \gamma[\gamma m + 2(2-\gamma)]\mathbb{J} - 2(2-\gamma)G + \frac{m}{n}G \\
&= (2-\gamma)^2\mathbb{I} + \gamma(2-\gamma)\mathbb{J} - (2-\gamma)\big[2 - \frac{m}{n(2-\gamma)}\big]G \\
&= (2-\gamma)\big[(2-\gamma)\mathbb{I} + \gamma\mathbb{J} - (2 - \frac{m}{n(2-\gamma)})G\big].
\end{aligned}$$

Therefore, $G'^2 = \frac{m}{n}G'$ if and only if $2 - \gamma = m/n$ if and only if $m = 2n + 1$. The moreover part follows from Theorem 2.6. In a similar way we can prove second part of the theorem as well. $\square$

For proving the perturbation result for regular k-distance frame, we need the following result which is given by Casazza and Christensen ([18]).

Theorem 2.8. *Let $X = \{x_j\}_{j\in J}$ be a frame for a Hilbert space H with frame bounds C and D. Assume that $\{y_j\}_{j\in J}$ is a sequence of H and that there exist $\lambda_1, \lambda_2, \mu \geq 0$ such that* $\max\{\lambda_1 + \frac{\mu}{\sqrt{C}}, \lambda_2\} < 1$. *Suppose one of the following conditions holds for any finite scalar sequence $\{c_j\}_{j\in J}$ and every $x \in H$. Then $\{y_j\}_{j\in J}$ is also a frame for H.*
(i) $(\sum_{j\in J} |\langle x, x_j - y_j\rangle|^2)^{1/2} \leq \lambda_1(\sum_{j\in J} |\langle x, x_j\rangle|^2)^{1/2} + \lambda_2(\sum_{j\in J} |\langle x, y_j\rangle|^2)^{1/2} + \mu\|x\|$;
(ii) $\|\sum_{j=1}^{n} c_j(x_j - y_j)\| \leq \lambda_1\|\sum_{j=1}^{n} c_j x_j\| + \lambda_2\|\sum_{j=1}^{n} c_j y_j\| + \mu(\sum_{j=1}^{n} |c_j|^2)^{1/2}$.
Moreover, if $\{x_j\}_{j\in J}$ is a Riesz basis for H and $\{y_j\}_{j\in J}$ satisfies (ii), then $\{y_j\}_{j\in J}$ is also a Riesz basis for H.

Theorem 2.9. *Let $X = \{x_j\}_{j\in J}$ be a regular k-distance frame for a Hilbert space H with frame bounds C and D. Assume that $\{y_j\}_{j\in J}$ is a sequence of H such that $\{y_j\}_{j\in J} = T\{x_j\}_{j\in J}$ where T is unitary, and that there exist $\lambda_1, \lambda_2, \mu \geq 0$ such that $\max\Big\{\lambda_1 + \dfrac{\mu}{\sqrt{C}}, \lambda_2\Big\} < 1$. Suppose one of the following conditions holds for any finite scalar sequence $\{c_j\}_{j\in J}$ and every $x \in H$. Then $\{y_j\}_{j\in J}$ is also a regular k-distance frame for H.*

(i) $(\sum\limits_{j\in J} |\langle x, x_j - y_j\rangle|^2)^{1/2} \leq \lambda_1 (\sum\limits_{j\in J} |\langle x, x_j\rangle|^2)^{1/2} + \lambda_2 (\sum\limits_{j\in J} |\langle x, y_j\rangle|^2)^{1/2} + \mu\|x\|$;

(ii) $\|\sum\limits_{j=1}^{n} c_j(x_j - y_j)\| \leq \lambda_1 \|\sum\limits_{j=1}^{n} c_j x_j\| + \lambda_2 \|\sum\limits_{j=1}^{n} c_j y_j\| + \mu(\sum\limits_{j=1}^{n} |c_j|^2)^{1/2}$.

Proof. By using Theorem 2.8, we can say that $\{y_j\}_{j\in J}$ is a frame for H if one of the conditions holds for any finite scalar sequence $\{c_j\}_{j\in J}$ and every $x \in H$. Since $\{y_j\}_{j\in J} = T\{x_j\}_{j\in J}$ where T is unitary, one can easily verify that $\{y_j\}_{j\in J}$ is a regular k-distance set as it is given that $\{x_j\}_{j\in J}$ is a regular k-distance set. Thus, $\{y_j\}_{j\in J}$ is also a regular k-distance frame for H. $\square$

Example 2.4. *Consider a regular three-distance frame as given*

$$\{x_j\}_{j=1}^{4} = \Big\{(0,1), (0,-1), \Big(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Big), \Big(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\Big)\Big\}$$

with lower and upper bound $C = 1$ and $D = 3$, respectively. Now let

$$\{y_j\}_{j=1}^{4} = \Big\{\Big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Big), \Big(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\Big), (-1,0), (1,0)\Big\}$$

and $\lambda_1 = 0.36$, $\lambda_2 = 0.80$ and $\mu = 0$.

Then $\{y_j\}_{j=1}^{4}$ is a regular three-distance frame with lower and upper frame bound 0.6 *and* 3.4*, respectively.*

2.3 Conclusions

Frame theory in Hilbert space has emerged as a significant tool for immense applications in science and engineering. In this chapter, we have introduced the concept of a regular k-distance frame in Hilbert space. Here, we discussed various characteristics of regular k-distance sets as well as focused on k-distance tight frames for the underlying space. We also studied them from an operator theoretic approach and discussed the dual frames for regular k-distance sets and provide some examples. In the end, we established a perturbation result for regular k-distance frames.

CHAPTER 3

Weaving g-frames in Hilbert C^*-modules

Woven frames are motivated by distributed signal processing with potential applications in wireless sensor networks. g-frames provide more choices for analyzing functions from the frame expansion coefficients. This chapter aims to introduce woven g-frames in Hilbert C^*-modules, and to develop their fundamental properties. In this investigation, we establish sufficient conditions under which two g-frames possess the weaving properties. We also investigate the sufficient conditions under which a family of g-frames possesses weaving properties.

3.1 Introduction and Preliminaries

Sun [53] introduced the concept of g-frame or generalized frames in Hilbert spaces. A. Khosravi and B. Khosravi [45] defined g-frame in Hilbert C^*-module. Weaving frames are powerful tools in wireless sensor networks and pre-processing signals. Bemrose et al. [6] introduced weaving frames in Hilbert space, and fundamental properties of woven frames were developed.

Now we recall some basic definitions from the literature.

Let X and Y be separable Hilbert spaces, and $\{Y_j : j \in J\}$ be a sequence of closed subspaces of Y. Let $\mathcal{L}(X, Y_j)$ be the collection of all bounded linear operators from X into Y.

Definition 3.1. *[53] We call a sequence $\{\Lambda_j \in \mathcal{L}(X, Y_j) \colon j \in J\}$ a generalized frame, or simply a g-frame, for X with respect to $\{Y_j : j \in J\}$ if there are two positive constants A*

and B such that

$$A\|x\|^2 \leq \sum_{j\in J} \|\Lambda_j x\|^2 \leq B\|x\|^2, \ \forall\, x \in X. \tag{3.1}$$

Definition 3.2. *[6] Let I be a countable indexing set. A family of frames* $\{\{\phi_{ij}\}_{j\in I} : i \in [m]\}$ *for H is said to be woven, if there are universal constants A and B such that for every partition* $\{\sigma_i\}_{i\in[m]}$ *of I, the family* $\bigcup_{i\in[m]} \{\phi_{ij}\}_{j\in\sigma_i}$ *is a frame for H with frame bounds A and B.*

Definition 3.3. *[47] A family of g-frames* $\{\Lambda_{ij}\}_{i\in I, j\in[m]}$ *for a Hilbert space H is said to be woven if there are universal constants A and B so that for every partition* $\{\sigma_j\}_{j\in[m]}$ *of I, the family* $\{\Lambda_{ij}\}_{i\in\sigma_j, j\in[m]}$ *is a g-frame for H with lower and upper frame bounds A and B, respectively.*

Let $\mathcal{U}$ and $\mathcal{V}$ be finitely or countably generated Hilbert $\mathcal{A}$-modules, and $\{\mathcal{V}_i : i \in I\}$ be a sequence of closed Hilbert submodules of $\mathcal{V}$. Let $End^*_{\mathcal{A}}(\mathcal{U}, \mathcal{V}_i)$ be the collection of all adjointable $\mathcal{A}$-linear maps from $\mathcal{U}$ to $\mathcal{V}_i$.

Definition 3.4. *[45] A sequence* $\{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{U}, \mathcal{V}_i) \colon i \in I\}$ *is called a g-frame or a generalized frame in* $\mathcal{U}$ *with respect to* $\{\mathcal{V}_i \colon i \in I\}$ *if there exist constants* $C, D > 0$ *such that for every* $f \in \mathcal{U}$*,*

$$C\langle f, f\rangle \leq \sum_{i\in I} \langle \Lambda_i f, \Lambda_i f\rangle \leq D\langle f, f\rangle. \tag{3.2}$$

Definition 3.5. *[34] A family* $\{\{\Lambda_{ij}\}_{i\in I}\}_{j\in J}$ *of frames for* $\mathcal{H}$ *is called woven if there exist universal constants* $0 < A < B < \infty$ *such that for every partition* $\{\sigma_j\}_{j\in J}$ *of I, the family* $\{\{\Lambda_{ij}\}_{i\in I}\}_{j\in J}$ *is a frame for* $\mathcal{H}$ *with lower and upper frame bounds A and B, respectively. Each family* $\{\{\Lambda_{ij}\}_{i\in\sigma_i}\}_{j\in J}$ *is called a weaving.*

We now give an example of woven frames in Hilbert C^*-module.

Example 3.1. *Let* ℓ^∞ *be the unitary* C^**-algebra of all bounded complex-valued sequences with the following operations*

$$uv = \{u_i v_i\}_{i\in\mathbb{N}},\ u^* = \{\overline{u_i}\}_{i\in\mathbb{N}},\ \forall\, u = \{u_i\}_{i\in\mathbb{N}}, v = \{v_i\}_{i\in\mathbb{N}} \in \ell^\infty.$$

Let $\mathcal{H} = C_0$ be the set of all sequences converging to zero. Then C_0 is a Hilbert ℓ^∞-module with ℓ^∞-valued inner product

$$\langle u, v \rangle = uv^* = \{u_i v_i^*\}_{i\in\mathbb{N}} = \{u_i \overline{v_i}\}_{i\in\mathbb{N}} \ \forall \ u, v \in C_0$$

Let $J = \mathbb{N}$ and we define $\Lambda = \{\Lambda_{1j}\}_{j=1}^\infty \in \mathcal{H}$ and $\Gamma = \{\Lambda_{2j}\}_{j=1}^\infty \in \mathcal{H}$ as follows:

$$\begin{aligned} \{\Lambda_{1j}\}_{j=1}^\infty &= \{e_1, e_2, 0, e_3, 0, e_4, 0, e_5, ...\} \\ \{\Lambda_{2j}\}_{j=1}^\infty &= \{0, e_2, e_2, e_3, e_3, e_4, e_4, e_5, e_5, ...\} \end{aligned}$$

where $\{e_j\}_{j=1}^\infty$ be the standard orthonormal basis for $\mathcal{H}$.
Let $f = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, ...\} \in \mathcal{H}$. Then $\langle f, f \rangle = \{\alpha_1\alpha_1^*, \alpha_2\alpha_2^*, \alpha_3\alpha_3^*, \alpha_4\alpha_4^*, ...\}$. Here partial ordering $' \le '$ means pointwise comparision.
For any subset σ of $\mathbb{N}$, we have

$$\sum_{j\in\sigma}\langle f, \Lambda_{1j}\rangle\langle\Lambda_{1j}, f\rangle + \sum_{j\in\sigma^c}\langle f, \Lambda_{2j}\rangle\langle\Lambda_{2j}, f\rangle \le 2\sum_{j=1}^\infty\langle f, e_j\rangle\langle e_j, f\rangle = 2\langle f, f\rangle.$$

On the other hand, let $f \in \mathcal{H}$. Then we have

$$\begin{aligned} \langle f, f\rangle &= \sum_{j=1}^\infty\langle f, e_j\rangle\langle e_j, f\rangle \\ &\le \sum_{j\in\sigma}\langle f, \Lambda_{1j}\rangle\langle\Lambda_{1j}, f\rangle + \sum_{j\in\sigma^c}\langle f, \Lambda_{2j}\rangle\langle\Lambda_{2j}, f\rangle. \end{aligned}$$

Hence Λ and Γ are woven frames with universal lower and upper frame bounds 1 *and* 2*, respectively.*

3.2 Main Results

The above literature motivates us to introduce the notion of weaving g-frames in Hilbert C^*-modules.

Definition 3.6. *Two g-frames $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Gamma = \{\Gamma_i\}_{i\in I}$ for $\mathcal{U}$ are said to be g-woven if there exist universal positive constants A and B such that for any partition σ of $I = \mathbb{N}$,* *the family* $\{\Lambda_i\}_{i\in\sigma} \bigcup \{\Gamma_i\}_{i\in\sigma^c}$ *is a g-frame for* $\mathcal{U}$ *with lower and upper g-frame bounds* A *and* B*, respectively i.e.*

$$A\langle f,f\rangle \leq \sum_{i\in\sigma}\langle \Lambda_i f, \Lambda_i f\rangle + \sum_{i\in\sigma^c}\langle \Gamma_i f, \Gamma_i f\rangle \leq B\langle f,f\rangle,\ \forall\ f\in\mathcal{U}.$$

Definition 3.7. *A family of g-frames* $\{\{\Lambda_{ij}\}_{j=1}^{\infty} : i\in I\}$ *for* $\mathcal{U}$ *with respect to* $\{\mathcal{V}_i : i\in I\}$ *is said to be g-woven if there exist universal positive constants* A *and* B *such that for any partition* $\{\sigma_i\}_{i\in I}$ *of* $\mathbb{N}$*, the family* $\bigcup\limits_{i\in I}\{\Lambda_{ij}\}_{j\in\sigma_i}$ *is a g-frame for* $\mathcal{U}$ *with lower and upper g-frame bounds* A *and* B*, respectively.*

Let $\{\mathcal{V}_i : i\in I\}$ be a sequence of Hilbert $\mathcal{A}$-modules, we define the space

$$\bigoplus_{i\in I}\mathcal{V}_i = \{\{c_{ij}\}_{j\in\sigma_i, i\in I} : c_{ij}\in\mathcal{V}_i \text{ and } \sum_{j\in\sigma_i, i\in I}\langle c_{ij}, c_{ij}\rangle \text{is norm convergent in } \mathcal{A}\}$$

with the inner product defined by

$$\langle\{c_{ij}\},\{d_{ij}\}\rangle_{i\in I, j\in\sigma_i} = \sum_{i\in I}\sum_{j\in\sigma_i}\langle c_{ij}, d_{ij}\rangle.$$

Associated with a woven g-frame $\{\{\Lambda_{ij}\}_{j=1}^{\infty} : i\in I\}$, we define the analysis, synthesis, and frame operator as follows:
The operator $T\colon\mathcal{U}\to\bigoplus\limits_{i\in I}\mathcal{V}_i$ defined by

$$Tf = \{\Lambda_{ij} f\}_{i\in I, j\in\sigma_i}$$

is called the *analysis operator*.
The *synthesis operator* $T^*\colon \bigoplus\limits_{i\in I}\mathcal{V}_i\to\mathcal{U}$ is given by

$$T^*\{c_{ij}\}_{i\in I, j\in\sigma_i} = \sum_{i\in I}\sum_{j\in\sigma_i}\Lambda_{ij}^* c_{ij}.$$

By composing T and T^*, we obtain the *frame operator* $S\colon \mathcal{U} \to \mathcal{U}$ as

$$\begin{aligned} Sf &= T^*Tf \\ &= \sum_{i\in I}\sum_{j\in\sigma_i} \Lambda_{ij}^*\Lambda_{ij}f, \end{aligned}$$

where Λ_{ij}^* is the adjoint operator of Λ_{ij}.

Proposition 3.1. *Let $\{\{\Lambda_{ij}\}_{j=1}^{\infty} : i \in I\}$ be g- woven frame for $\mathcal{U}$ with universal bounds A and B. Then the frame operator S is self adjoint, positive, bounded and invertible on $\mathcal{U}$.*

Proof. Since $S^* = (T^*T)^* = T^*T = S$, the frame operator S is self adjoint.
Let $\{\{\Lambda_{ij}\}_{j=1}^{\infty} : i \in I\}$ be woven g-frame for $\mathcal{U}$ with universal lower and upper frame bounds A and B, respectively.
Let $f \in \mathcal{U}$ and $Sf = \sum\limits_{i\in I}\sum\limits_{j\in\sigma_i} \Lambda_{ij}^*\Lambda_{ij}f$ then

$$\begin{aligned} \langle Sf, f\rangle &= \langle \sum_{i\in I}\sum_{j\in\sigma_i} \Lambda_{ij}^*\Lambda_{ij}f, f\rangle \\ &= \sum_{i\in I}\sum_{j\in\sigma_i} \langle \Lambda_{ij}f, \Lambda_{ij}f\rangle. \end{aligned}$$

$\implies A\langle f, f\rangle \leq \langle Sf, f\rangle \leq B\langle f, f\rangle$
$\implies AI \leq S \leq BI.$
Therefore, the frame operator S is positive, bounded and invertible. □

Theorem 3.1. *Let $\{\{\Lambda_{ij}\}_{j=1}^{\infty} : i \in I\}$ be a g-Bessel sequence for $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i \in I\}$ and with g-Bessel bounds B_j. Then, every weaving is a g-Bessel sequence with bound $\sum\limits_{j=1}^{m} B_j$.*

Proof. Let $\{\sigma_j\}_{j\in J}$ be any partition of I. Then for every $f \in \mathcal{H}$, we have

$$\begin{aligned} \sum_{j=1}^{m}\sum_{i\in\sigma_j} \langle \Lambda_{ij}f, \Lambda_{ij}f\rangle &\leq \sum_{j=1}^{m}\sum_{i\in I} \langle \Lambda_{ij}f, \Lambda_{ij}f\rangle \\ &\leq \sum_{j=1}^{m} B_j\langle f, f\rangle. \end{aligned}$$

□

Proposition 3.2. *Let $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ and $\Gamma = \{\Gamma_i\}_{i\in\mathbb{N}}$ be g-Bessel sequences in $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i \in \mathbb{N}\}$ with g-Bessel bounds B_1, B_2, respectively. If $J \subset \mathbb{N}$, and $\Lambda_J \equiv \{\Lambda_j\}_{j\in J}$ and $\Gamma_J \equiv \{\Gamma_j\}_{j\in J}$ are woven g-frames, then Λ and Γ are woven g-frames for $\mathcal{U}$.*

Proof. Let A be lower g-frame bound for Λ_J and Γ_J, and let $\sigma \subset \mathbb{N}$ be an arbitrary subset. Then,

$$\begin{aligned}
A\langle f, f\rangle &\leq \sum_{j\in\sigma\cap J}\langle\Lambda_j f, \Lambda_j f\rangle + \sum_{j\in\sigma^c\cap J}\langle\Gamma_j f, \Gamma_j f\rangle \\
&\leq \sum_{j\in\sigma}\langle\Lambda_j f, \Lambda_j f\rangle + \sum_{j\in\sigma^c}\langle\Gamma_j f, \Gamma_j f\rangle \\
&\leq (B_1 + B_2)\langle f, f\rangle.
\end{aligned}$$

Hence, Λ and Γ are woven g-frames for $\mathcal{U}$. □

Theorem 3.2. *Let $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ and $\Gamma = \{\Gamma_i\}_{i\in\mathbb{N}}$ be g-woven frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i \in I\}$ with universal g-frame bounds A and B. If $J \subset \mathbb{N}$ and*

$$\sum_{j\in J}\langle\Lambda_j f, \Lambda_j f\rangle \leq D\langle f, f\rangle$$

for all $f \in \mathcal{U}$ and for some $0 < D < A$, then $\Lambda_0 \equiv \{\Lambda_i\}_{i\in\mathbb{N}\setminus J}$ and $\Gamma_0 \equiv \{\Gamma_i\}_{i\in\mathbb{N}\setminus J}$ are g-woven frames for $\mathcal{U}$ with universal g-frame bounds $A - D$ and B.

Proof. Let σ be any subset of $\mathbb{N}\setminus J$. We compute

$$\begin{aligned}
&\sum_{j\in\sigma}\langle\Lambda_j f, \Lambda_j f\rangle + \sum_{j\in(\mathbb{N}\setminus J)\setminus\sigma}\langle\Gamma_j f, \Gamma_j f\rangle \\
&\qquad = \Big(\sum_{j\in\sigma\bigcup J}\langle\Lambda_j f, \Lambda_j f\rangle - \sum_{j\in J}\langle\Lambda_j f, \Lambda_j f\rangle\Big) + \sum_{j\in(\mathbb{N}\setminus J)\setminus\sigma}\langle\Gamma_j f, \Gamma_j f\rangle \\
&\qquad = \Big(\sum_{j\in\sigma\bigcup J}\langle\Lambda_j f, \Lambda_j f\rangle + \sum_{j\in(\mathbb{N}\setminus J)\setminus\sigma}\langle\Gamma_j f, \Gamma_j f\rangle\Big) - \sum_{j\in J}\langle\Lambda_j f, \Lambda_j f\rangle \\
&\qquad \geq (A - D)\langle f, f\rangle.
\end{aligned}$$

On the other hand, for all $f \in \mathcal{U}$, we have

$$\begin{aligned}\sum_{j\in\sigma}\langle\Lambda_j f,\Lambda_j f\rangle + \sum_{j\in(\mathbb{N}\setminus J)\setminus\sigma}\langle\Gamma_j f,\Gamma_j f\rangle &\leq \sum_{j\in\sigma\bigcup J}\langle\Lambda_j f,\Lambda_j f\rangle + \sum_{j\in(\mathbb{N}\setminus J)\setminus\sigma}\langle\Gamma_j f,\Gamma_j f\rangle\\ &\leq B\langle f,f\rangle.\end{aligned}$$

Hence, Λ_0 and Γ_0 are g-woven frames for $\mathcal{U}$ with the universal lower and upper g-frame bounds $A-D$ and B, respectively. $\square$

We require the following lemmas to prove our results.

Lemma 3.1. *[1] Let $\mathcal{A}$ be a C^*-algebra. Let $\mathcal{U}$ and $\mathcal{V}$ be two Hilbert $\mathcal{A}$-modules and $T \in End^*_{\mathcal{A}}(\mathcal{U},\mathcal{V})$. Then the following statements are equivalent:*

1. *T is surjective.*

2. *T^* is bounded below with respect to norm i.e there exists $m>0$ such that $\|T^*f\| \geq m\|f\|$ for all $f \in \mathcal{U}$.*

3. *T^* is bounded below with respect to inner product i.e there exists $m>0$ such that $\langle T^*f,T^*f\rangle \geq m\langle f,f\rangle$ for all $f\in\mathcal{U}$.*

Lemma 3.2. *[50] Let $\mathcal{U}$ and $\mathcal{V}$ be Hilbert $\mathcal{A}$-modules over a C^*-algebra $\mathcal{A}$ and let $T:\mathcal{U}\to\mathcal{V}$ be a linear map. Then the following conditions are equivalent:*

1. *The operator T is bounded and $\mathcal{A}$-linear.*

2. *There exists $k \geq 0$ such that $\langle Tx,Tx\rangle \leq k\langle x,x\rangle$ holds for all $x\in\mathcal{U}$.*

Theorem 3.3. *Let $\Lambda=\{\Lambda_i\}_{i\in\mathbb{N}}$ and $\Gamma=\{\Gamma_i\}_{i\in\mathbb{N}}$ be a family of g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i\in\mathbb{N}\}$. Then for every partition σ of $\mathbb{N}$, Λ and Γ are g-woven frames for $\mathcal{U}$ with the universal lower and upper g-frame bounds A and B, respectively if and only if*

$$A\|\langle f,f\rangle\| \leq \|\sum_{i\in\sigma}\langle\Lambda_i f,\Lambda_i f\rangle + \sum_{i\in\sigma^c}\langle\Gamma_i f,\Gamma_i f\rangle\| \leq B\|\langle f,f\rangle\|$$

for all $f\in\mathcal{U}$.

Proof. ($\Longrightarrow$) Obvious.

Now we assume that there exist constants $0 < A, B < \infty$ such that for all $f \in \mathcal{U}$

$$A\|\langle f, f\rangle\| \leq \|\sum_{i\in\sigma}\langle \Lambda_i f, \Lambda_i f\rangle + \sum_{i\in\sigma^c}\langle \Gamma_i f, \Gamma_i f\rangle\| \leq B\|\langle f, f\rangle\|. \tag{3.3}$$

We prove that Λ and Γ are g-woven frames for $\mathcal{U}$ with the universal lower and upper g-frame bounds A and B, respectively.

As S is positive, self adjoint and invertible operator. We have

$$\langle S^{\frac{1}{2}} f, S^{\frac{1}{2}} f\rangle = \langle Sf, f\rangle = \sum_{i\in\sigma}\langle \Lambda_i f, \Lambda_i f\rangle + \sum_{i\in\sigma^c}\langle \Gamma_i f, \Gamma_i f\rangle.$$

From equation (3.3), we have

$$\sqrt{A}\|f\| \leq \|S^{\frac{1}{2}} f\| \leq \sqrt{B}\|f\|.$$

By using Lemma 3.1, we have

$$\langle S^{\frac{1}{2}} f, S^{\frac{1}{2}} f\rangle = \langle Sf, f\rangle \geq A\langle f, f\rangle.$$

Since $S^{\frac{1}{2}}$ is bounded and $\mathcal{A}$-linear, by using Lemma 3.2, we have

$$\langle S^{\frac{1}{2}} f, S^{\frac{1}{2}} f\rangle = \langle Sf, f\rangle \leq B\langle f, f\rangle.$$

From the above two inequalities we conclude that Λ and Γ are g-woven frames for $\mathcal{U}$ with the universal lower and upper g-frame bounds A and B, respectively.

$\square$

Theorem 3.4. *Let $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ and $\Gamma = \{\Gamma_i\}_{i\in\mathbb{N}}$ be g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i \in \mathbb{N}\}$ with g-frame bounds A_1, B_1 and A_2, B_2, respectively. Assume that there are constants $0 < \lambda_1, \lambda_2, \mu < 1$ such that*

$$\lambda_1\sqrt{B_1} + \lambda_2\sqrt{B_2} + \mu \leq \frac{A_1}{2(\sqrt{B_1} + \sqrt{B_2})}$$

and

$$\begin{aligned}
&\|\sum_{i\in\mathbb{N}}\langle(\Lambda_i^*-\Gamma_i^*)f_i,(\Lambda_i^*-\Gamma_i^*)f_i\rangle\|^{\frac{1}{2}}\\
&\leq \lambda_1\|\sum_{i\in\mathbb{N}}\langle\Lambda_i^* f_i,\Lambda_i^* f_i\rangle\|^{\frac{1}{2}}+\lambda_2\|\sum_{i\in\mathbb{N}}\langle\Gamma_i^* f_i,\Gamma_i^* f_i\rangle\|^{\frac{1}{2}}+\mu\|\langle\{f_i\},\{f_i\}\rangle\|^{\frac{1}{2}} \qquad (3.4)
\end{aligned}$$

for all $\{f_i\}_{i\in\mathbb{N}} \in \big(\bigoplus_{i\in\mathbb{N}} V_i\big)$*. Then,* Λ *and* Γ *are g-woven frames with universal lower and upper g-frame bounds* $\frac{A_1}{2}$*,* B_1+B_2*, respectively.*

Proof. Let T and R be the synthesis operator for the frames $\{\Lambda_i\}_{i\in\mathbb{N}}$ and $\{\Gamma_i\}_{i\in\mathbb{N}}$, respectively defined as follows $T\colon \bigoplus_{i\in\mathbb{N}} \mathcal{V}_i \to \mathcal{U}$ is given by

$$T\{f_i\} = \sum_{i\in\mathbb{N}} \Lambda_i^* f_i$$

and $R\colon \bigoplus_{i\in\mathbb{N}} \mathcal{V}_i \to \mathcal{U}$ is given by

$$R\{f_i\} = \sum_{i\in\mathbb{N}} \Gamma_i^* f_i$$

For each $\sigma \subset \mathbb{N}$, define bounded operators

$$T_\sigma, R_\sigma\colon (\bigoplus_{i\in\mathbb{N}} V_i) \to \mathcal{U}$$

$T_\sigma(\{f_i\}) = \sum_{i\in\sigma} \Lambda_i^* f_i$ and $R_\sigma(\{f_i\}) = \sum_{i\in\sigma} \Gamma_i^* f_i$.
We note that $\|T_\sigma\| \leq \|T\|$, $\|R_\sigma\| \leq \|R\|$ and $\|T_\sigma - R_\sigma\| \leq \|T-R\|$.
As we know $\|f\|^2 = \|\langle f,f\rangle\|, \forall f \in \mathcal{U}$ and using equation (3.4), we have

$$\begin{aligned}
\lambda_1\|T(\{f_i\}_{i\in\mathbb{N}})\|+\lambda_2\|R(\{f_i\}_{i\in\mathbb{N}})\|+\mu\|\{f_i\}_{i\in\mathbb{N}}\| &\geq \|\sum_{i\in\mathbb{N}}\langle(\Lambda_i^*-\Gamma_i^*)f_i,(\Lambda_i^*-\Gamma_i^*)f_i\rangle\|^{\frac{1}{2}}\\
&= \|(T-R)(\{f_i\}_{i\in\mathbb{N}})\|
\end{aligned}$$

This gives $\|T-R\| \leq \lambda_1\|T\|+\lambda_2\|R\|+\mu$.

Using this, for any $\sigma \subset \mathbb{N}$, we compute

$$\begin{aligned}
\|\sum_{i\in\sigma}\Lambda_i^*\Lambda_i f-\sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\| &= \|T_\sigma(\{\Lambda_i f\})\}_{i\in\sigma}-R_\sigma(\{\Gamma_i f\})\}_{i\in\sigma}\| \\
&= \|T_\sigma T_\sigma^* f-R_\sigma R_\sigma^* f\| \\
&= \|T_\sigma T_\sigma^* f-T_\sigma R_\sigma^* f+T_\sigma R_\sigma^* f-R_\sigma R_\sigma^* f\| \\
&\leq \|(T_\sigma T_\sigma^*-T_\sigma R_\sigma^*)f\|+\|(T_\sigma R_\sigma^*-R_\sigma R_\sigma^*)f\| \\
&\leq \|T_\sigma\|\|T_\sigma^*-R_\sigma^*\|\|f\|+\|T_\sigma-R_\sigma\|\|R_\sigma^*\|\|f\| \\
&\leq \|T\|\|T-R\|\|f\|+\|T-R\|\|R\|\|f\| \\
&\leq (\lambda_1\|T\|+\lambda_2\|R\|+\mu)(\|T\|+\|R\|)\|f\| \\
&\leq (\lambda_1\|T\|+\lambda_2\|R\|+\mu)(\sqrt{B_1}+\sqrt{B_2})\|f\| \\
&< \frac{A_1}{2(\sqrt{B_1}+\sqrt{B_2})}(\sqrt{B_1}+\sqrt{B_2})\|f\| \\
&= \frac{A_1}{2}\|f\|. \qquad (3.5)
\end{aligned}$$

Now, by using equation (3.5), it follows that

$$\begin{aligned}
\|\sum_{i\in\sigma^c}\Lambda_i^*\Lambda_i f+\sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\| &= \|\sum_{i\in\sigma^c}\Lambda_i^*\Lambda_i f+\sum_{i\in\sigma}\Lambda_i^*\Lambda_i f-\sum_{i\in\sigma}\Lambda_i^*\Lambda_i f+\sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\| \\
&= \|\sum_{i\in\mathbb{N}}\Lambda_i^*\Lambda_i f+\sum_{i\in\sigma}\Gamma_i^*\Gamma_i f-\sum_{i\in\sigma}\Lambda_i^*\Lambda_i f\| \\
&\geq \|\sum_{i\in\mathbb{N}}\Lambda_i^*\Lambda_i f\|-\|\sum_{i\in\sigma}\Lambda_i^*\Lambda_i f-\sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\| \\
&\geq A_1\|f\|-\|\sum_{i\in\sigma}\Lambda_i^*\Lambda_i f-\sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\| \\
&\geq A_1\|f\|-\frac{A_1}{2}\|f\| \\
&= \frac{A_1}{2}\|f\|.
\end{aligned}$$

This gives universal lower g-frame bound. By using Theorem 3.1, we get B_1+B_2 as universal upper g-frame bound. Hence, Λ and Γ are g-woven frames with universal lower and upper g-frame bounds $\frac{A_1}{2}$, B_1+B_2, respectively. . □

Theorem 3.5. *Let $\Lambda=\{\Lambda_i\}_{i\in\mathbb{N}}$ and $\Gamma=\{\Gamma_i\}_{i\in\mathbb{N}}$ be g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i\in\mathbb{N}\}$ with g-frame bounds A_1, B_1 and A_2, B_2, respectively. Assume that there are* *constants* $0 < \lambda, \mu, \gamma < 1$ *such that*

$$\lambda B_1 + \mu B_2 + \gamma < A_1$$

and

$$\begin{aligned}
&\| \sum_{i\in\sigma} \langle (\Lambda_i^*\Lambda_i - \Gamma_i^*\Gamma_i)f, (\Lambda_i^*\Lambda_i - \Gamma_i^*\Gamma_i)f \rangle \|^{\frac{1}{2}} \\
&\quad \leq \lambda \| \sum_{i\in\sigma} \langle \Lambda_i^*\Lambda_i f, \Lambda_i^*\Lambda_i f \rangle \|^{\frac{1}{2}} + \mu \| \sum_{i\in\sigma} \langle \Gamma_i^*\Gamma_i f, \Gamma_i^*\Gamma_i f \rangle \|^{\frac{1}{2}} + \gamma (\sum_{i\in\sigma} \|\Lambda_i f\|^2)^{\frac{1}{2}} \qquad (3.6)
\end{aligned}$$

for all $f \in \mathcal{U}$ *and for every* $\sigma \subset \mathbb{N}$. *Then,* Λ *and* Γ *are g-woven frames with universal g-frame bounds* $(A_1 - \lambda B_1 - \mu B_2 - \gamma\sqrt{B_1})$ *and* $(B_1 + \lambda B_1 + \mu B_2 + \gamma\sqrt{B_1})$.

Proof. For any $\sigma \subset \mathbb{N}$, we use the fact that for $f \in \mathcal{U}$,
$\| \sum_{i\in\sigma} \Lambda_i^*\Lambda_i f\| \leq B_1\|f\|$ and $\| \sum_{i\in\sigma} \Gamma_i^*\Gamma_i f\| \leq B_2\|f\|$
and as we know that $\|f\|^2 = \|\langle f, f\rangle\|, \forall f \in \mathcal{U}$, (3.6) implies

$$\begin{aligned}
\| \sum_{i\in\sigma} (\Lambda_i^*\Lambda_i - \Gamma_i^*\Gamma_i)f\| &\leq \lambda \| \sum_{i\in\sigma} \Lambda_i^*\Lambda_i f\| + \mu \| \sum_{i\in\sigma} \Gamma_i^*\Gamma_i f\| \\
&+ \gamma (\sum_{i\in\sigma} \|\Lambda_i f\|^2)^{\frac{1}{2}} \qquad (3.7)
\end{aligned}$$

We compute

$$\begin{aligned}
\| \sum_{i\in\sigma^c} \Lambda_i^*\Lambda_i + \sum_{i\in\sigma} \Gamma_i^*\Gamma_i f\| &= \| \sum_{i\in\mathbb{N}} \Lambda_i^*\Lambda_i + \sum_{i\in\sigma} \Gamma_i^*\Gamma_i f - \sum_{i\in\sigma} \Lambda_i^*\Lambda_i f\| \\
&\geq \| \sum_{i\in\mathbb{N}} \Lambda_i^*\Lambda_i f\| - \| \sum_{i\in\sigma} \Gamma_i^*\Gamma_i f - \sum_{i\in\sigma} \Lambda_i^*\Lambda_i f\| \\
&\geq A_1\|f\| - \| \sum_{i\in\sigma} \Gamma_i^*\Gamma_i f - \sum_{i\in\sigma} \Lambda_i^*\Lambda_i f\| \\
&\geq A_1\|f\| - \lambda \| \sum_{i\in\sigma} \Lambda_i^*\Lambda_i f\| - \mu \| \sum_{i\in\sigma} \Gamma_i^*\Gamma_i f\| - \gamma (\sum_{i\in\sigma} \|\Lambda_i f\|^2)^{\frac{1}{2}} \\
&\geq (A_1 - \lambda B_1 - \mu B_2 - \gamma\sqrt{B_1})\|f\|
\end{aligned}$$

and

$$\begin{aligned}\|\sum_{i\in\sigma^c}\Lambda_i^*\Lambda_i f+\sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\| &= \|\sum_{i\in\mathbb{N}}\Lambda_i^*\Lambda_i f+\sum_{i\in\sigma}\Gamma_i^*\Gamma_i f-\sum_{i\in\sigma}\Lambda_i^*\Lambda_i f\|\\ &\leq \|\sum_{i\in\mathbb{N}}\Lambda_i^*\Lambda_i f\|+\|\sum_{i\in\sigma}\Gamma_i^*\Gamma_i f-\sum_{i\in\sigma}\Lambda_i^*\Lambda_i f\|\\ &\leq B_1\|f\|+\lambda\|\sum_{i\in\sigma}\Lambda_i^*\Lambda_i f\|+\mu\|\sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\|+\gamma(\sum_{i\in\sigma}\|\Lambda_i f\|^2)^{\frac{1}{2}}\\ &\leq (B_1+\lambda B_1+\mu B_2+\gamma\sqrt{B_1})\|f\|.\end{aligned}$$

Therefore, Λ and Γ are g-woven frames with the universal lower and upper bounds $(A_1-\lambda B_1-\mu B_2-\gamma\sqrt{B_1})$ and $(B_1+\lambda B_1+\mu B_2+\gamma\sqrt{B_1})$, respectively. □

Theorem 3.6. *For $i\in I$, let $\Lambda_i=\{\Lambda_{ij}\}_{j\in J}$ be a family of g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i\in I\}$ with bounds A_i and B_i. For any $\sigma\subset J$ and a fix $t\in I$, let $P_i^{\sigma}(f)=\sum_{j\in\sigma}\Lambda_{ij}^*\Lambda_{ij}f-\sum_{j\in\sigma}\Lambda_{tj}^*\Lambda_{tj}f$ for $i\neq t$. If P_i^{σ} is a positive linear operator, then the family of g-frames $\{\Lambda_i\}_{i\in I}$ is g-woven.*

Proof. Let $\{\sigma_i\}_{i\in I}$ be any partition of J. Then, for every $f\in\mathcal{U}$, a fix $t\in I$ and $j\in\sigma_i$, we have

$$\begin{aligned}\sum_{j\in\sigma_i}\langle\Lambda_{tj}^*\Lambda_{tj}f,f\rangle &= \sum_{j\in\sigma_i}\left\langle\Lambda_{ij}^*\Lambda_{ij}f-P_i^{\sigma}(f),f\right\rangle\\ &\leq \sum_{j\in\sigma_i}\left\langle\Lambda_{ij}^*\Lambda_{ij}f,f\right\rangle \text{ (As } P_i^{\sigma} \text{ is a positive linear operator)}\end{aligned}\tag{3.8}$$

Now,

$$\begin{aligned}A_t\langle f,f\rangle &\leq \sum_{j\in J}\langle\Lambda_{ij}^*\Lambda_{ij}f,f\rangle\\ &= \sum_{j\in\sigma_1}\langle\Lambda_{tj}^*\Lambda_{tj}f,f\rangle+...+\sum_{j\in\sigma_i}\langle\Lambda_{tj}^*\Lambda_{tj}f,f\rangle+...+\sum_{j\in\sigma_m}\langle\Lambda_{tj}^*\Lambda_{tj}f,f\rangle\\ &\leq \sum_{j\in\sigma_1}\langle\Lambda_{1j}^*\Lambda_{1j}f,f\rangle+...+\sum_{j\in\sigma_i}\langle\Lambda_{ij}^*\Lambda_{ij}f,f\rangle+...+\sum_{j\in\sigma_m}\langle\Lambda_{mj}^*\Lambda_{mj}f,f\rangle \quad \text{(Using (3.8))}\\ &\leq (B_1+...+B_i+...+B_m)\langle f,f\rangle\\ &= \sum_{i\in I}B_i\langle f,f\rangle\end{aligned}$$

which implies

$$A_t\langle f,f\rangle \leq \sum_{i\in I}\sum_{j\in\sigma_i}\langle \Lambda_{ij}^*\Lambda_{ij}f,f\rangle \leq \sum_{i\in I} B_i\langle f,f\rangle.$$

□

Theorem 3.7. *For each $j \in [m]$, let $\Lambda_i = \{\Lambda_{ij}\}_{i\in I}$ be a family of g-frame for $\mathcal{U}$ with bounds A_j and B_j. Suppose there exists $K > 0$ such that*

$$\sum_{i\in J}\|\langle(\Lambda_{ij}-\Lambda_{il})f,(\Lambda_{ij}-\Lambda_{il})f\rangle\| \leq K\min\Big\{\sum_{i\in J}\|\langle\Lambda_{ij}f,\Lambda_{ij}f\rangle\|,\sum_{i\in J}\|\langle\Lambda_{il}f,\Lambda_{il}f\rangle\|\Big\}, \quad (j,l\in[m], j\neq l)$$

for all $f \in \mathcal{U}$ and for all subsets $J \subset I$. Then the family of g-frames $\{\{\Lambda_{ij}\}_{i\in I} : j \in [m]\}$ is woven with universal frame bounds $\dfrac{\sum_{j\in[m]} A_j}{2(m-1)(K+1)+1}$ and $\sum_{j\in[m]} B_j$.

Proof. Let $\{\sigma_j\}_{j\in[m]}$ be any partition of I. For the lower frame inequality, we have'

$$
\begin{aligned}
&\sum_{j\in[m]} A_j\|\langle f,f\rangle\| \\
&= A_1\|\langle f,f\rangle\|+...+A_m\|\langle f,f\rangle\| \\
&\leq \sum_{i\in I}\|\langle \Lambda_{i1}f,\Lambda_{i1}f\rangle\|+...+\sum_{i\in I}\|\langle \Lambda_{im}f,\Lambda_{im}f\rangle\| \\
&= \big(\sum_{i\in\sigma_1}\|\langle \Lambda_{i1}f,\Lambda_{i1}f\rangle\|+...+\sum_{i\in\sigma_m}\|\langle \Lambda_{i1}f,\Lambda_{i1}f\rangle\|\big)+... \\
&\quad +\big(\sum_{i\in\sigma_1}\|\langle \Lambda_{im}f,\Lambda_{im}f\rangle\|+...+\sum_{i\in\sigma_m}\|\langle \Lambda_{im}f,\Lambda_{im}f\rangle\|\big) \\
&\leq \big[\sum_{i\in\sigma_1}\|\langle \Lambda_{i1}f,\Lambda_{i1}f\rangle\|+2\big(\sum_{i\in\sigma_2}\|\langle(\Lambda_{i1}-\Lambda_{i2})f,(\Lambda_{i1}-\Lambda_{i2})f\rangle\|+\sum_{i\in\sigma_2}\|\langle \Lambda_{i2}f,\Lambda_{i2}f\rangle\|\big)+... \\
&\quad +2\big(\sum_{i\in\sigma_m}\|\langle(\Lambda_{i1}-\Lambda_{im})f,(\Lambda_{i1}-\Lambda_{im})f\rangle\|+\sum_{i\in\sigma_m}\|\langle \Lambda_{im}f,\Lambda_{im}f\rangle\|\big)\big]+... \\
&\quad +\big[2\big(\sum_{i\in\sigma_1}\|\langle(\Lambda_{im}-\Lambda_{i1})f,(\Lambda_{im}-\Lambda_{i1})f\rangle\|+\sum_{i\in\sigma_1}\|\langle \Lambda_{i1}f,\Lambda_{i1}f\rangle\|\big)+... \\
&\quad +2\big(\sum_{i\in\sigma_{m-1}}\|\langle(\Lambda_{im}-\Lambda_{i(m-1)})f,(\Lambda_{im}-\Lambda_{i(m-1)})f\rangle\|+\sum_{i\in\sigma_m}\|\langle \Lambda_{i(m-1)}f,\Lambda_{i(m-1)}f\rangle\|\big) \\
&\quad +\sum_{i\in\sigma_m}\|\langle \Lambda_{im}f,\Lambda_{im}f\rangle\|\big] \\
&\leq \big[\sum_{i\in\sigma_1}\|\langle \Lambda_{i1}f,\Lambda_{i1}f\rangle\|+2\big(K\sum_{i\in\sigma_2}\|\langle \Lambda_{i2}f,\Lambda_{i2}f\rangle\|+\sum_{i\in\sigma_2}\|\langle \Lambda_{i2}f,\Lambda_{i2}f\rangle\|\big)+... \\
&\quad +2\big(K\sum_{i\in\sigma_m}\|\langle \Lambda_{im}f,\Lambda_{im}f\rangle\|+\sum_{i\in\sigma_m}\|\langle \Lambda_{im}f,\Lambda_{im}f\rangle\|\big)\big]+... \\
&\quad +\big[2\big(K\sum_{i\in\sigma_1}\|\langle \Lambda_{i1}f,\Lambda_{i1}f\rangle\|+\sum_{i\in\sigma_1}\|\langle \Lambda_{i1}f,\Lambda_{i1}f\rangle\|\big)+... \\
&\quad +2\big(K\sum_{i\in\sigma_{m-1}}\|\langle \Lambda_{i(m-1)}f,\Lambda_{i(m-1)})f\rangle\|+\sum_{i\in\sigma_{(m-1)}}\|\langle \Lambda_{i(m-1)}f,\Lambda_{i(m-1)}f\rangle\|\big) \\
&\quad +\sum_{i\in\sigma_m}\|\langle \Lambda_{im}f,\Lambda_{im}f\rangle\|\big] \\
&= \sum_{i\in\sigma_1}\|\langle \Lambda_{i1}f,\Lambda_{i1}f\rangle\|+...+\sum_{i\in\sigma_m}\|\langle \Lambda_{im}f,\Lambda_{im}f\rangle\| \\
&\qquad +(m-1)2(K+1)\Big(\sum_{i\in\sigma_1}\|\langle \Lambda_{i1}f,\Lambda_{i1}f\rangle\|+...+\sum_{i\in\sigma_m}\|\langle \Lambda_{im}f,\Lambda_{im}f\rangle\|\Big) \\
&= \Big[2(m-1)(K+1)+1\Big]\sum_{j\in[m]}\sum_{i\in\sigma_j}\|\langle \Lambda_{ij}f,\Lambda_{ij}f\rangle\|
\end{aligned}
$$

for all $f\in\mathcal{U}$. From Proposition 4.1, we know that $\{\{\Lambda_{ij}\}_{i\in I}: j\in[m]\}$ satisfies upper frame inequality with universal upper frame bound $\sum_{j\in[m]} B_j$. Hence, for all

$f \in \mathcal{U}$, we have

$$\frac{\sum_{j\in[m]} A_j}{2(m-1)(K+1)+1}\|\langle f,f\rangle\| \leq \sum_{j\in[m]}\sum_{i\in\sigma_j}\|\langle \Lambda_{ij}f,\Lambda_{ij}f\rangle\| \leq \sum_{j\in[m]} B_j\|\langle f,f\rangle\|.$$

The proof is completed. □

Proposition 3.3. *Let $\{\Lambda_{ij}\}_{i\in I,j\in[m]}$ be a family of woven g-Bessel sequence for $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i \in I\}$ and with g-Bessel with bound B. Then, $\{\Lambda_{ij}T\}_{i\in I,j\in[m]}$ is also woven g-Bessel sequence with bound $B\|T\|^2$ for every $T \in L(\mathcal{U})$.*

Proof. Suppose $\{\Lambda_{ij}\}_{i\in I,j\in[m]}$ be a family of woven g-Bessel sequence for $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i \in I\}$ and with g-Bessel bound B. Then for any partiotion $\{\sigma_j\}_{j\in[m]}$ of I, we have

$$\sum_{j=1}^{m}\sum_{i\in\sigma_j}\langle \Lambda_{ij}f,\Lambda_{ij}f\rangle \quad\leq\quad B\langle f,f\rangle.$$

Now

$$\begin{aligned}\sum_{j=1}^{m}\sum_{i\in\sigma_j}\langle \Lambda_{ij}Tf,\Lambda_{ij}Tf\rangle &\leq B\langle Tf,Tf\rangle\\ &\leq B\|T\|^2\langle f,f\rangle.\end{aligned}$$

□

Theorem 3.8. *Let $\{\Lambda_{ij}\}_{i\in I,j\in[m]}$ be a family of g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i \in I\}$. Then $\{\Lambda_{ij}\}_{i\in I,j\in[m]}$ is a woven g-Bessel sequence with bound D if and only if*

$$\|\sum_{j=1}^{m}\sum_{i\in\sigma_j}\langle \Lambda_{ij}f,\Lambda_{ij}f\rangle\| \leq D\|f\|^2,\ \forall\ f \in \mathcal{U}$$

holds for any partition $\{\sigma_j\}_{j\in[m]}$ of I.

Proof. ($\Longrightarrow$) Obvious

On the other hand, we define a linear operator $T\colon \mathcal{U} \to \bigoplus\limits_{i\in I} \mathcal{V}_i$ by

$$Tf \;=\; \sum_{j=1}^{m}\sum_{i\in\sigma_j} \Lambda_{ij} f e_{ij}$$

for any partiotion $\{\sigma_j\}_{j\in[m]}$ of I, where $\{e_{ij}\}_{i\in\sigma_j, j\in[m]}$ are the standard orthonormal basis for $\mathcal{V}_i$.

Then

$$\|Tf\|^2 = \|\langle Tf, Tf\rangle\| = \|\sum_{j=1}^{m}\sum_{i\in\sigma_j}\langle \Lambda_{ij}f, \Lambda_{ij}f\rangle\| \le D\|f\|^2$$

which implies that $\|Tf\| \le \sqrt{D}\|f\|$. Hence T is bounded. It is obvious that T is $\mathcal{A}$-linear. Then by Lemma 3.2, we have

$$\langle Tf, Tf\rangle \le D\langle f, f\rangle.$$

Equivalently, $\sum\limits_{j=1}^{m}\sum\limits_{i\in\sigma_j}\langle \Lambda_{ij}f, \Lambda_{ij}f\rangle \le D\langle f, f\rangle$, as desired. □

Example 3.2. *Let $\mathcal{A} = \ell^\infty, \mathcal{U} = C_0$ the Hilbert $\mathcal{A}$-module of the set of all null sequences equipped with the $\mathcal{A}$-inner product*

$$\langle u, v\rangle = uv^* = \{u_i v_i^*\}_{i=1}^\infty = \{u_i\overline{v_i}\}_{i=1}^\infty$$

for any $u = \{u_i\}_{i=1}^\infty \in \mathcal{U}$ and $v = \{v_i\}_{i=1}^\infty \in \mathcal{U}$.

Let $j \in J = \mathbb{N}$ and define $A_j \in B(\mathcal{U})$ by $A_j\{f_i\}_{i\in\mathbb{N}} = \{\delta_{ij}f_j\}_{i\in\mathbb{N}},\ \forall\{f_i\}_{i\in\mathbb{N}} \in \mathcal{U}$.

Let $\Lambda = \{\Lambda_j\}_{j=1}^\infty$ and $\Gamma = \{\Gamma_j\}_{j=1}^\infty$ be defined as follows:

$$\begin{aligned}\{\Lambda_j\}_{j=1}^\infty &= \{A_1 + A_2, A_1 + A_2, 0, 0, 0, ...\}\\ \{\Gamma_j\}_{j=1}^\infty &= \{0, 0, A_3, A_4, A_5, ...\}\end{aligned}$$

Let $f = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, ...\} \in \mathcal{H}$. Then $\langle f, f\rangle = \{\alpha_1\alpha_1^*, \alpha_2\alpha_2^*, \alpha_3\alpha_3^*, \alpha_4\alpha_4^*, ...\}$. Here partial ordering $'\le'$ means pointwise comparision.

For any subset σ of $\mathbb{N}$, we have

$$\sum_{j\in\sigma}\langle\Lambda_j f,\Lambda_j f\rangle+\sum_{j\in\sigma^c}\langle\Gamma_j f,\Gamma_j f\rangle\leq 2\langle f,f\rangle.$$

On the other hand, it is clear that

$$\langle f,f\rangle \;\;\leq\;\; \sum_{j\in\sigma}\langle\Lambda_j f,\Lambda_j f\rangle+\sum_{j\in\sigma^c}\langle\Gamma_j f,\Gamma_j f\rangle.$$

Hence Λ and Γ are woven g-frames with universal lower and upper frame bounds 1 and 2, respectively.

Theorem 3.9. *Let $\Lambda=\{\Lambda_i\}_{i\in\mathbb{N}}$ be a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i\in\mathbb{N}\}$ with upper and lower g-frame bounds A and B, respectively. Suppose S is the g-frame operator of Λ_i such that $S^{-1}\Lambda_i$ is self adjoint for all $i\in\mathbb{N}$. Then $\{\Lambda_i\}_{i\in\mathbb{N}}$ and $\{\Lambda_i^*S^{-1}\}_{i\in\mathbb{N}}$ are woven g-frames for $\mathcal{U}$.*

Proof. Let σ be any partition of $\mathbb{N}$. Since S^{-1} and $S^{-1}\Lambda_i$ are self adjoint, we have

$$\begin{aligned}
A\langle f,f\rangle &= \sum_{i\in\mathbb{N}}\langle\Lambda_i f,\Lambda_i f\rangle\\
&= \sum_{i\in\sigma}\langle\Lambda_i f,\Lambda_i f\rangle+\sum_{i\in\sigma^c}\langle\Lambda_i f,\Lambda_i f\rangle\\
&= \sum_{i\in\sigma}\langle\Lambda_i f,\Lambda_i f\rangle+\sum_{i\in\sigma^c}\langle SS^{-1}\Lambda_i f,SS^{-1}\Lambda_i f\rangle\\
&\leq \sum_{i\in\sigma}\langle\Lambda_i f,\Lambda_i f\rangle+\sum_{i\in\sigma^c}\|S\|^2\langle S^{-1}\Lambda_i f,S^{-1}\Lambda_i f\rangle\\
&\leq \sum_{i\in\sigma}\langle\Lambda_i f,\Lambda_i f\rangle+B^2\sum_{i\in\sigma^c}\langle(S^{-1}\Lambda_i)^*f,(S^{-1}\Lambda_i)^*f\rangle\\
&= \sum_{i\in\sigma}\langle\Lambda_i f,\Lambda_i f\rangle+B^2\sum_{i\in\sigma^c}\langle\Lambda_i^*S^{-1}f,\Lambda_i^*S^{-1}f\rangle\\
&\leq \max\{1,B^2\}\big(\sum_{i\in\sigma}\langle\Lambda_i f,\Lambda_i f\rangle+\sum_{i\in\sigma^c}\langle\Lambda_i^*S^{-1}f,\Lambda_i^*S^{-1}f\rangle\big).
\end{aligned}$$

Thus, $\min\{A,\dfrac{A}{B^2}\}$is a universal lower g-frame bound. To find a universal upper

g-frame bound, we compute

$$\begin{aligned}
\sum_{i\in\sigma}\langle\Lambda_i f,\Lambda_i f\rangle+\sum_{i\in\sigma^c}\langle\Lambda_i^* S^{-1}f,\Lambda_i^* S^{-1}f\rangle &= \sum_{i\in\sigma}\langle\Lambda_i f,\Lambda_i f\rangle+\sum_{i\in\sigma^c}\langle(S^{-1}\Lambda_i)^* f,(S^{-1}\Lambda_i)^* f\rangle\\
&= \sum_{i\in\sigma}\langle\Lambda_i f,\Lambda_i f\rangle+\sum_{i\in\sigma^c}\langle S^{-1}\Lambda_i f,S^{-1}\Lambda_i f\rangle\\
&\leq \sum_{i\in\sigma}\langle\Lambda_i f,\Lambda_i f\rangle+\sum_{i\in\sigma^c}\|S^{-1}\|^2\langle\Lambda_i f,\Lambda_i f\rangle\\
&\leq \sum_{i\in\sigma}\langle\Lambda_i f,\Lambda_i f\rangle+\frac{1}{A^2}\sum_{i\in\sigma^c}\langle\Lambda_i f,\Lambda_i f\rangle\\
&\leq \max\{1,\frac{1}{A^2}\}\sum_{i\in\mathbb{N}}\langle\Lambda_i f,\Lambda_i f\rangle\\
&\leq B\max\{1,\frac{1}{A^2}\}\langle f,f\rangle.
\end{aligned}$$

Hence, $\{\Lambda_i\}_{i\in\mathbb{N}}$ and $\{\Lambda_i^* S^{-1}\}_{i\in\mathbb{N}}$ are woven g-frames for $\mathcal{U}$ with universal lower g-frame bound $\min\{A,\frac{A}{B^2}\}$ and universal upper g-frame bound $\max\{B,\frac{B}{A^2}\}$. $\square$

3.3 Conclusions

In this chapter, we extended the concept of weaving frames to weaving g-frames in Hilbert C^*-modules and defined woven g-frames in Hilbert C^*-modules and developed its fundamental properties. We established sufficient conditions under which two g-frames possess the weaving properties. We also investigated the sufficient conditions under which a family of g-frames possesses weaving properties. We also established the equivalent definition for woven g-frames in Hilbert C^*-modules.

CHAPTER 4

Weaving K-frames in Hilbert C^*-modules

In [34], F. Ghobadzadeh *et al.* studied and investigated various fundamental properties of weaving frames in Hilbert C^*-module. As K-frames and standard frames differ in many aspects, we introduce the concept of weaving K-frames and an atomic system for weaving K-frames in Hilbert C^*-module. In this chapter, we study weaving K-frames from an operator theoretic point of view. We give an equivalent definition for weaving K-frames and characterize weaving K-frames in terms of bounded linear operators. We also investigate the invariance of woven Bessel sequence under an adjointable operator.

4.1 Introduction and Preliminaries

Deepshikha and Lalit K. Vashisht [55] studied weaving properties of K-frames in Hilbert space and presented necessary and sufficient conditions for weaving K-frames in Hilbert space. They have also shown that woven K-frames and weakly woven K-frames are equivalent.

Now we recall some basic definitions from the literature.

Definition 4.1. *[49] A sequence $\{\psi_j\}_{j\in J}$ of elements in a Hilbert $\mathcal{A}$-module $\mathcal{H}$ is said to be a K-frame $(K \in L(\mathcal{H}))$ if there exist constants $C, D > 0$ such that*

$$C\langle K^* f, K^* f\rangle \leq \sum_{j\in J}\langle f, \psi_j\rangle\langle \psi_j, f\rangle \leq D\langle f, f\rangle,\ \forall\ f \in \mathcal{H}. \tag{4.1}$$

Definition 4.2. *[55] A family of K-frames $\{\{\phi_{ij}\}_{j\in I} : i \in [m]\}$ for H is said to be K-woven if there exist universal positive constants A and B such that for any partition*

$\{\sigma_i\}_{i\in[m]}$ *of* $\mathbb{N}$, *the family* $\bigcup\limits_{i\in[m]} \{\phi_{ij}\}_{j\in\sigma_i}$ *is a K-frame for H with lower and upper K-frame bounds A and B, respectively. Each family* $\bigcup\limits_{i\in[m]} \{\phi_{ij}\}_{j\in\sigma_i}$ *is called a weaving.*

Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of real or complex numbers.

4.2 Main Results

We define weaving K-frame in Hilbert C^*-modules.

Definition 4.3. *Let* $\mathcal{H}$ *be a Hilbert* $\mathcal{A}$*-module over a unital* C^**-algebra. A family of K-frames* $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ *for* $\mathcal{H}$ *is said to be K-woven if there exist universal positive constants A and B such that for any partition* $\{\sigma_i\}_{i\in I}$ *of* $\mathbb{N}$, *the family* $\bigcup\limits_{i\in I}\{f_{ij}\}_{j\in\sigma_i}$ *is a K-frame for* $\mathcal{H}$ *with lower and upper K-frame bounds A and B, respectively. Each family* $\bigcup\limits_{i\in I}\{f_{ij}\}_{j\in\sigma_i}$ *is called a weaving.*

The woven frame is called tight woven frame if $A = B$ and it is called normalized woven tight frame if $A = B = 1$.

For any partition $\{\sigma_i\}_{i\in I}$ of $\mathbb{N}$, we define the space as

$$\bigoplus_{i\in I} \ell^2(\sigma_i) = \{\{c_{ij}\}_{j\in\sigma_i, i\in I} | c_{ij} \in \mathcal{A}, \sum_{i\in I}\sum_{j\in\sigma_i} c_{ij}c_{ij}^* \text{ converges in } \|\cdot\|_{\mathcal{A}}\}$$

with the inner product

$$\langle\{c_{ij}\}_{j\in\sigma_i, i\in I}, \{d_{ij}\}_{j\in\sigma_i, i\in I}\rangle = \sum_{i\in I}\sum_{j\in\sigma_i} c_{ij}d_{ij}^*.$$

Let the family of K-frames $\{F_i = \{f_{ij}\}_{j\in J} : i \in I\}$ be woven for $\mathcal{H}$, for any partition $\{\sigma_i\}_{i\in I}$ of J and $W = \{f_{ij}\}_{j\in\sigma_i, i\in I}$ be a K-frame for $\mathcal{H}$, then we have the corresponding synthesis, analysis, and frame operator as follows:

The operator $T_W \colon \bigoplus_{i\in I} \ell^2(\sigma_i) \to \mathcal{H}$ defined by

$$\begin{aligned} T_W(\{c_{ij}\})_{i\in I, j\in\sigma_i} &= \sum_{i\in I} T_{F_i} D_{\sigma_i}(\{c_{ij}\}) \\ &= \sum_{i\in I}\sum_{j\in\sigma_i} c_{ij} f_{ij} \end{aligned} \tag{4.2}$$

is called the *synthesis or pre-frame operator*, where T_{F_i} is the synthesis operator of F_i and D_{σ_i} is a $|J| \times |J|$ diagonal matrix with $d_{jj} = 1$ for $j \in \sigma_i$ and otherwise 0.
The adjoint of T_W is

$$\begin{aligned} \langle f, T_W\{c_{ij}\}\rangle_{i\in I, j\in\sigma_i} &= \langle f, \sum_{i\in I}\sum_{j\in\sigma_i} c_{ij} f_{ij}\rangle \\ &= \sum_{i\in I}\sum_{j\in\sigma_i} c_{ij}^* \langle f, f_{ij}\rangle. \end{aligned} \tag{4.3}$$

$\implies \langle f, T_W\{c_{ij}\}\rangle = \langle\{\langle f, f_{ij}\rangle\}, \{c_{ij}\}\rangle$
$\implies T_W^*(f) = \{\langle f, f_{ij}\rangle\}_{i\in I, j\in\sigma_i}$.
The adjoint operator $T_W^* \colon \mathcal{H} \to \bigoplus_{i\in I} \ell^2(\sigma_i)$ is given by

$$\begin{aligned} T_W^*(f) &= \sum_{i\in I} D_{\sigma_i} T_{F_i}^{\sigma_i *}(f) \\ &= \{\langle f, f_{ij}\rangle\}_{i\in I, j\in\sigma_i} \end{aligned} \tag{4.4}$$

and is called the *analysis operator*.
By composing T_W and T_W^*, we obtain the *frame operator* $S_W \colon \mathcal{H} \to \mathcal{H}$

$$\begin{aligned} S_W(f) &= T_W T_W^*(f) \\ &= (\sum_{i\in I} T_{F_i} D_{\sigma_i})(\sum_{i\in I} T_{F_i} D_{\sigma_i})^* \\ &= \sum_{i\in I}\sum_{j\in\sigma_i} \langle f, f_{ij}\rangle f_{ij}. \end{aligned} \tag{4.5}$$

We now state some of the important properties of the synthesis, analysis, and frame operator of weaving K-frames in Hilbert C^*-module.

Lemma 4.1. *Let $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ be a woven Bessel sequence then the synthesis operator T_W is linear and bounded.*

Proof. Let $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ be a woven Bessel sequence with universal Bessel bound B.

Now,

$$\begin{aligned} T_W(\{\lambda c_{ij} + d_{ij}\}) &= \sum_{i\in I}\sum_{j\in\sigma_i}(\lambda c_{ij} + d_{ij})f_{ij} \\ &= \sum_{i\in I}\sum_{j\in\sigma_i}\lambda c_{ij}f_{ij} + \sum_{i\in I}\sum_{j\in\sigma_i} d_{ij}f_{ij} \\ &= \lambda T_W(\{c_{ij}\}) + T_W(\{d_{ij}\}). \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} \|T_W f\|^2 &= \|\langle T_W f, T_W f\rangle\| \\ &= \|\langle T_W T_W^* f, f\rangle\| \\ &= \|\langle S_W f, f\rangle\| \\ &\leq B\|f\|^2. \end{aligned} \tag{4.7}$$

$\implies \|T_W f\| \leq \sqrt{B}\|f\|$.

Hence, the synthesis operator T_W is linear and bounded. □

Lemma 4.2. *Let $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ be woven frame for $\mathcal{H}$ with universal bounds A and B. Then the frame operator S_W is self adjoint, positive, bounded and invertible on $\mathcal{H}$.*

Proof. Since $S_W^* = (T_W T_W^*)^* = T_W T_W^* = S_W$, the frame operator S_W is self adjoint. Let $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ be woven frame for $\mathcal{H}$ with universal bounds A and B. Let $f \in \mathcal{H}$ and $S_W(f) = \sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle f_{ij}$ then

$$\begin{aligned} \langle S_W f, f\rangle &= \Big\langle \sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle f_{ij}, f\Big\rangle \\ &= \sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle. \end{aligned}$$

$\implies A\langle f, f\rangle \leq \langle S_W f, f\rangle \leq B\langle f, f\rangle$

$\implies AI \leq S_W \leq BI$.

Therefore, the frame operator S_W is positive, bounded and invertible. □

We now give an example of woven K-frames in Hilbert C^*-module.

Example 4.1. *Let $\mathcal{H} = C_0$ be the set of all sequences converging to zero and let K be the orthogonal projection of $\mathcal{H}$ onto span$\{e_j\}_{j=3}^{\infty}$. For any $u = \{u_j\}_{j=1}^{\infty} \in \mathcal{H}$ and $v = \{v_j\}_{j=1}^{\infty} \in \mathcal{H}$,*

$$\langle u, v\rangle = uv^* = \{u_j v_j^*\}_{j=1}^{\infty}.$$

Let $\phi = \{\phi_{1j}\}_{j=1}^{\infty} \in \mathcal{H}$ and $\psi = \{\phi_{2j}\}_{j=1}^{\infty} \in \mathcal{H}$ be defined as follows:

$$\begin{aligned}\{\phi_{1j}\}_{j=1}^{\infty} &= \{0, e_3, 0, e_4, 0, e_5, 0, e_6, ...\}\\ \{\phi_{2j}\}_{j=1}^{\infty} &= \{0, e_3, e_3, e_4, e_4, e_5, e_5, e_6, e_6, ...\},\end{aligned}$$

where $\{e_j\}_{j=1}^{\infty}$ be the standard orthonormal basis for $\mathcal{H}$.
Let $f = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, ...\} \in \mathcal{H}$. Then $\langle f, f\rangle = \{\alpha_1\alpha_1^*, \alpha_2\alpha_2^*, \alpha_3\alpha_3^*, \alpha_4\alpha_4^*, ...\}$
For any subset σ of $\mathbb{N}$, we have

$$\sum_{j\in\sigma}\langle f, \phi_{1j}\rangle\langle\phi_{1j}, f\rangle + \sum_{j\in\sigma^c}\langle f, \phi_{2j}\rangle\langle\phi_{2j}, f\rangle \leq 2\sum_{j=1}^{\infty}\langle f, e_j\rangle\langle e_j, f\rangle = 2\langle f, f\rangle.$$

On the other hand, let $f \in \mathcal{H}$. Then $f = \sum\limits_{j=1}^{\infty}\alpha_j e_j$. Thus, we have

$$\begin{aligned}\langle K^*f, K^*f\rangle &= \big\langle K^*(\sum_{j=1}^{\infty}\alpha_j e_j), K^*(\sum_{j=1}^{\infty}\alpha_j e_j)\big\rangle\\ &= \big\langle \sum_{j=3}^{\infty}\alpha_j e_j, \sum_{j=3}^{\infty}\alpha_j e_j\big\rangle\\ &= \sum_{j=3}^{\infty}\langle f, e_j\rangle\langle e_j, f\rangle\\ &\leq \sum_{j\in\sigma}\langle f, \phi_{1j}\rangle\langle\phi_{1j}, f\rangle + \sum_{j\in\sigma^c}\langle f, \phi_{2j}\rangle\langle\phi_{2j}, f\rangle.\end{aligned}$$

Hence ϕ and ψ are K-woven frames with universal lower and upper frame bounds 1 *and* 2*, respectively.*

We now introduce a woven atomic system for weaving K-frames in Hilbert C^*-module.

Definition 4.4. *The sequence $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ of $\mathcal{H}$ is said to be a woven atomic system for $K \in L(\mathcal{H})$, if for any partition $\{\sigma_i\}_{i \in I}$ of $\mathbb{N}$, the family $\bigcup_{i \in I}\{f_{ij}\}_{j \in \sigma_i}$ is a woven atomic system for K, i.e. the following statements hold:*
(i) The series $\sum_{i \in I}\sum_{j \in \sigma_i} c_{ij} f_{ij}$ converges for all $\{c_{ij}\}_{j \in \sigma_i, i \in I} \in \ell^2(\mathcal{A})$.
(ii) There exist $C > 0$ such that for every $f \in \mathcal{H}$, there exists $\{a_{ij,f}\}_{j \in \sigma_i, i \in I} \in \ell^2(\mathcal{A})$ such that $\sum_{i \in I}\sum_{j \in \sigma_i} a_{ij,f} a_{ij,f}^* \leq C\langle f, f\rangle$ and $Kf = \sum_{i \in I}\sum_{j \in \sigma_i} a_{ij,f} f_{ij}$.

Theorem 4.1. *If $K \in L(\mathcal{H})$, then there exists a woven atomic system for K.*

Proof. Let $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ be a standard normalized woven tight frame for $\mathcal{H}$ with universal frame bound $A = B = 1$.
Since

$$f = \sum_{i \in I}\sum_{j \in \sigma_i} \langle f, f_{ij}\rangle f_{ij}$$

We have

$$Kf = \sum_{i \in I}\sum_{j \in \sigma_i} \langle f, f_{ij}\rangle K f_{ij}.$$

For $f \in \mathcal{H}$, $a_{ij,f} = \langle f, f_{ij}\rangle$ and $g_{ij} = Kf_{ij}$

$$\begin{aligned}
\sum_{i \in I}\sum_{j \in \sigma_i} \langle f, g_{ij}\rangle\langle g_{ij}, f\rangle &= \sum_{i \in I}\sum_{j \in \sigma_i} \langle f, Kf_{ij}\rangle\langle Kf_{ij}, f\rangle \\
&= \sum_{i \in I}\sum_{j \in \sigma_i} \langle K^* f, f_{ij}\rangle\langle f_{ij}, K^* f\rangle \\
&= \langle K^* f, K^* f\rangle \\
&\leq \|K^*\|^2\langle f, f\rangle.
\end{aligned}$$

Therefore, $\{\{g_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven Bessel sequence for $\mathcal{H}$ with Bessel bound $\|K^*\|^2$ and we conclude that the series $\sum_{i \in I}\sum_{j \in \sigma_i} c_{ij} g_{ij}$ converges for all $\{c_{ij}\}_{j \in \sigma_i, i \in I} \in \ell^2(\mathcal{A})$.

We also have

$$\begin{aligned}\sum_{i\in I}\sum_{j\in\sigma_i} a_{ij,f}a_{ij,f}^* &= \sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle \\ &= \langle f, f\rangle\end{aligned}$$

which completes the proof. $\square$

Since it is more convenient to work with an equivalent definition of weaving K-frames in Hilbert C^*-modules, we would like to introduce an equivalent definition in the following result. We quote the following results from the literature that will be used in our work.

Theorem 4.2. *[26] Let $\mathcal{F}, \mathcal{H}, \mathcal{K}$ be Hilbert C^*-modules over a C^*-algebra $\mathcal{A}$. Also let $S \in L(\mathcal{K}, \mathcal{H})$ and $T \in L(\mathcal{F}, \mathcal{H})$ with $\overline{R(T^*)}$ orthogonally complemented. The following statements are equivalent:*
(i) $SS^* \leq \lambda TT^*$ for some $\lambda > 0$;
(ii) there exists $\mu > 0$ such that $\|S^*z\| \leq \|T^*z\|$ for all $z \in \mathcal{H}$;
(iii) there exists $D \in L(\mathcal{K}, \mathcal{F})$ such that $S = TD$, i.e., $TX = S$ has a solution;
(iv) $R(S) \subseteq R(T)$.

Theorem 4.3. *For any partition $\{\sigma_i\}_{i\in I}$ of $\mathbb{N}$, let the family $\bigcup_{i\in I}\{f_{ij}\}_{j\in\sigma_i}$ be a woven Bessel sequence for $\mathcal{H}$ and $K \in L(\mathcal{H})$. Suppose that $T^* \in L(\mathcal{H}, \ell^2(\mathcal{A}))$ given by $T^*(f) = \{\langle f, f_{ij}\rangle\}_{i\in I, j\in\sigma_i}$ and $\overline{R(T^*)}$ is orthogonally complemented then the following statements are equivalent:*
(i) The sequence $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ of $\mathcal{H}$ is a woven atomic system for K.
(ii) There exist $A, B > 0$ such that

$$A\|K^*f\|^2 \leq \|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle\| \leq B\|f\|^2.$$

(iii) There exist $D \in L(\mathcal{H}, \ell^2(\mathcal{A}))$ such that $K = TD$.

Proof. (i) $\implies$ (ii) For every $f \in \mathcal{H}$, we have

$$\begin{aligned}\|K^* f\| &= \sup_{\|g\|=1} \|\langle g, K^* f\rangle\| \\ &= \sup_{\|g\|=1} \|\langle Kg, f\rangle\|\end{aligned}$$

Since $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven atomic system for K, there exist $C > 0$ such that for every $g \in \mathcal{H}$, there exist $a_g = \{a_{ij,g}\}_{j\in\sigma_i, i\in I} \in \ell^2(\mathcal{A})$ for which $\sum_{i\in I}\sum_{j\in\sigma_i} a_{ij,g}a^*_{ij,g} \leq C\langle g, g\rangle$ and $Kg = \sum_{i\in I}\sum_{j\in\sigma_i} a_{ij,g}f_{ij}$.

Therefore,

$$\begin{aligned}\|K^* f\|^2 &= \sup_{\|g\|=1} \|\langle Kg, f\rangle\|^2 \\ &= \sup_{\|g\|=1} \|\langle \sum_{i\in I}\sum_{j\in\sigma_i} a_{ij,g}f_{ij}, f\rangle\|^2 \\ &= \sup_{\|g\|=1} \|\sum_{i\in I}\sum_{j\in\sigma_i} a_{ij,g}\langle f_{ij}, f\rangle\|^2 \\ &\leq \sup_{\|g\|=1} \|\sum_{i\in I}\sum_{j\in\sigma_i} a_{ij,g}\|^2 \|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f_{ij}, f\rangle\|^2 \\ &= \sup_{\|g\|=1} \|\sum_{i\in I}\sum_{j\in\sigma_i} a_{ij,g}a^*_{ij,g}\| \|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle \langle f_{ij}, f\rangle\| \\ &\leq \sup_{\|g\|=1} C\|\langle g, g\rangle\| \|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle \langle f_{ij}, f\rangle\| \\ &= \sup_{\|g\|=1} C\|g\|^2 \|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle \langle f_{ij}, f\rangle\| \\ &= C\|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle \langle f_{ij}, f\rangle\| \end{aligned} \tag{4.8}$$

which implies $\frac{1}{C}\|K^* f\|^2 \leq \|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle \langle f_{ij}, f\rangle\|$.

Moreover, $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven Bessel sequence for $\mathcal{H}$. Hence (ii) holds.

(ii) $\implies$ (iii) Since $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven Bessel sequence for $\mathcal{H}$, we get

$$\begin{aligned}T(\{e_{ij}\}) &= \sum_{i\in I}\sum_{j\in\sigma_i} e_{ij}f_{ij} \\ &= f_{ij}\end{aligned}$$

where $\{e_{ij}\}_{j\in\sigma_i, i\in I}$ is the standard orthonormal basis for $\ell^2(\mathcal{A})$.

Therefore, for every $f \in \mathcal{H}$

$$\begin{aligned} A\|K^*f\|^2 &\leq \|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle \, \langle f_{ij}, f\rangle\| \\ &= \|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, T\{e_{ij}\}\rangle \, \langle T\{e_{ij}\}, f\rangle\| \\ &= \|\sum_{i\in I}\sum_{j\in\sigma_i}\langle T^*f, \{e_{ij}\}\rangle \, \langle\{e_{ij}\}, T^*f\rangle\| \\ &= \|T^*f\|^2. \end{aligned}$$

By using Theorem 4.2, there exist an operator $D \in L(\mathcal{H}, \ell^2(\mathcal{A}))$ such that $K = TD$.
(iii) $\implies$ (i) For every $f \in \mathcal{H}$, we have

$$\begin{aligned} Df &= \sum_{i\in I}\sum_{j\in\sigma_i}\langle Df, e_{ij}\rangle e_{ij} \\ \implies TDf &= \sum_{i\in I}\sum_{j\in\sigma_i}\langle Df, e_{ij}\rangle Te_{ij}. \end{aligned} \tag{4.9}$$

Let $a_{ij,f} = \langle Df, e_{ij}\rangle$, so for all $f \in \mathcal{H}$, we get

$$\begin{aligned} \sum_{i\in I}\sum_{j\in\sigma_i} a_{ij,f}a^*_{ij,f} &= \sum_{i\in I}\sum_{j\in\sigma_i}\langle Df, e_{ij}\rangle\langle e_{ij}, Df\rangle \\ &= \langle Df, Df\rangle \\ &\leq \|D\|^2\langle f, f\rangle. \end{aligned}$$

Since the sequence $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven Bessel sequence for $\mathcal{H}$, we conclude that $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven atomic system for K. □

Corollary 4.1. *Let $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ be a woven frame for $\mathcal{H}$ with universal frame bounds $A, B > 0$ and $K \in L(\mathcal{H})$. Then $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven atomic system for K with lower and upper frame bounds $\dfrac{1}{A^{-1}\|K\|^2}$ and B, respectively.*

Proof. Let S be the frame operator of $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$.
Since $\{\{S^{-1}f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven frame for $\mathcal{H}$ with bounds $B^{-1}, A^{-1} > 0$ and

$f = \sum_{i \in I} \sum_{j \in \sigma_i} \langle f, f_{ij} \rangle S^{-1} f_{ij}$, for all $f \in \mathcal{H}$.

$$\begin{aligned}
\|K^* f\|^2 &= \sup_{\|g\|=1} \|\langle K^* f, g \rangle\|^2 \\
&= \sup_{\|g\|=1} \|\langle \sum_{i \in I} \sum_{j \in \sigma_i} \langle f, f_{ij} \rangle K^* S^{-1} f_{ij}, g \rangle\|^2 \\
&= \sup_{\|g\|=1} \| \sum_{i \in I} \sum_{j \in \sigma_i} \langle f, f_{ij} \rangle \langle K^* S^{-1} f_{ij}, g \rangle\|^2 \\
&\leq \sup_{\|g\|=1} \| \sum_{i \in I} \sum_{j \in \sigma_i} \langle f, f_{ij} \rangle \langle f_{ij}, f \rangle\| \| \sum_{i \in I} \sum_{j \in \sigma_i} \langle Kg, S^{-1} f_{ij} \rangle \langle S^{-1} f_{ij}, Kg \rangle\| \\
&\leq \sup_{\|g\|=1} A^{-1} \| \sum_{i \in I} \sum_{j \in \sigma_i} \langle f, f_{ij} \rangle \langle f_{ij}, f \rangle\| \|Kg\|^2 \\
&\leq \sup_{\|g\|=1} A^{-1} \| \sum_{i \in I} \sum_{j \in \sigma_i} \langle f, f_{ij} \rangle \langle f_{ij}, f \rangle\| \|K\|^2 \|g\|^2 \\
&= A^{-1} \|K\|^2 \| \sum_{i \in I} \sum_{j \in \sigma_i} \langle f, f_{ij} \rangle \langle f_{ij}, f \rangle\|
\end{aligned}$$

which implies

$$\frac{1}{A^{-1}\|K\|^2} \|K^* f\|^2 \leq \| \sum_{i \in I} \sum_{j \in \sigma_i} \langle f, f_{ij} \rangle \langle f_{ij}, f \rangle\| \leq B\|f\|^2$$

and shows that the condition (ii) of Theorem 4.3 hold.
Therefore, $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven atomic system for K with lower and upper frame bounds $\dfrac{1}{A^{-1}\|K\|^2}$ and B, respectively. $\square$

Corollary 4.2. *Let $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ be a woven atomic system for K. If $K \in L(\mathcal{H})$ is onto, then $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven frame for $\mathcal{H}$.*

Proof. As we know, $K \in L(\mathcal{H})$ is surjective if and only if there exists $M > 0$ such that

$$M\|f\| \leq \|K^* f\|, \ \forall f \in \mathcal{H}. \tag{4.10}$$

Since $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven atomic system for K, so there exists $A, B > 0$ such that

$$A\|K^* f\|^2 \leq \| \sum_{i \in I} \sum_{j \in \sigma_i} \langle f, f_{ij} \rangle \langle f_{ij}, f \rangle\| \leq B\|f\|^2 \tag{4.11}$$

for any partition $\{\sigma_i\}_{i\in I}$ of $\mathbb{N}$.

By using (4.10) and (4.11), we get

$$AM^2\|f\|^2 \leq \|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle\| \leq B\|f\|^2$$

which completes the proof. $\square$

Proposition 4.1. *Let $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ be a family of K-frames for $\mathcal{H}$ with K-frame bounds A_i and B_i. Then, for any partition $\{\sigma_i\}_{i\in I}$ of $\mathbb{N}$, the family $\bigcup_{i\in I}\{f_{ij}\}_{j\in\sigma_i}$ is a woven Bessel sequence with Bessel bound $\sum_{i\in I} B_i$.*

Proof. Let $\{\sigma_i\}_{i\in I}$ be any partition of $\mathbb{N}$. Then, for any $f \in \mathcal{H}$

$$\begin{aligned}\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle &\leq \sum_{i\in I}\sum_{j\in\mathbb{N}}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle \\ &\leq \sum_{i\in I} B_i\langle f, f\rangle.\end{aligned}$$

$\square$

The following theorem gives a characterization of weaving K-frames in terms of a bounded linear operator in Hilbert C^*-module.

Theorem 4.4. *For each $i \in I$, suppose $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a family of K-frames for $\mathcal{H}$ with bounds A_i and B_i. Then the following conditions are equivalent:*

(i) The family $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is K-woven.

(ii) There exist $A > 0$ such that for any partition $\{\sigma_i\}_{i\in I}$ of $\mathbb{N}$, there exist a bounded linear operator $M_\sigma \colon l^2(\mathcal{A}) \to \mathcal{H}$ such that

$$M_\sigma(e_j) = \begin{cases} f_{1j}, & j \in \sigma_1 \\ f_{2j}, & j \in \sigma_2 \\ . & \\ . & \\ f_{mj}, & j \in \sigma_m \end{cases}$$

and $AKK^* \leq M_\sigma M_\sigma^*$, where $\{e_j\}_{j=1}^{\infty}$ is the standard orthonormal basis.

Proof. (i) $\implies$ (ii): Suppose A is a universal lower K-frame bound for the family $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$. For any partition $\{\sigma_i\}_{i\in I}$ of $\mathbb{N}$, let T_σ be the synthesis operator associated with the Bessel sequence $\bigcup_{i\in I}\{f_{ij}\}_{j\in\sigma_i}$.
Choose $M_\sigma = T_\sigma$.
Then $M_\sigma(e_j) = T_\sigma(e_j) = f_{ij},\ \forall\, i \in I, j \in \sigma_i$.
Now,

$$\begin{aligned}
A\langle K^*f, K^*f\rangle &= A\langle KK^*f, f\rangle \\
&\leq \sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle \\
&= \sum_{j\in\mathbb{N}}\langle f, M_\sigma(e_j)\rangle\langle M_\sigma(e_j), f\rangle \\
&= \sum_{j\in\mathbb{N}}\langle M_\sigma^*f, e_j\rangle\langle e_j, M_\sigma^*f\rangle \\
&= \Big\langle \sum_{j\in\mathbb{N}}\langle M_\sigma^*f, e_j\rangle e_j, M_\sigma^*f\Big\rangle \\
&= \langle M_\sigma^*f, M_\sigma^*f\rangle \\
&= \langle M_\sigma M_\sigma^*f, f\rangle.
\end{aligned}$$

This implies $AKK^* \leq M_\sigma M_\sigma^*$.
(ii) $\implies$ (i)
Let $\{\sigma_i\}_{i\in I}$ be any partition of $\mathbb{N}$.
Now,

$$\begin{aligned}
A\langle KK^*f, f\rangle &\leq \langle M_\sigma M_\sigma^*f, f\rangle \\
&= \langle M_\sigma^*f, M_\sigma^*f\rangle \\
&= \sum_{j\in\mathbb{N}}\langle M_\sigma^*f, e_j\rangle\langle e_j, M_\sigma^*f\rangle \\
&= \sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle.
\end{aligned}$$

This gives the universal lower K-frame bound A. And by Proposition 4.1, $\sum_{i\in I} B_i$ is one of the choice of an universal upper K-frame bound. □

In the following result, we investigate the invariance of the woven Bessel sequence under an adjointable operator.

Lemma 4.3. *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ be a family of woven Bessel sequences with universal Bessel bound D. Then $\{\{Mf_{ij}\}_{j=1}^{\infty} : i \in I\}$ are a family of woven Bessel sequences with universal Bessel bound with $D\|M^*\|^2$ for every $M \in L(\mathcal{H})$.*

Proof. Suppose $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ are woven Bessel sequences with universal Bessel bound D, then we have

$$\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle \leq D\langle f, f\rangle$$

for any partition $\{\sigma_i\}_{i\in I}$ of $\mathbb{N}$.
Then for any $f \in \mathcal{H}$,

$$\begin{aligned}\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, Mf_{ij}\rangle\langle Mf_{ij}, f\rangle &= \sum_{i\in I}\sum_{j\in\sigma_i}\langle M^*f, f_{ij}\rangle\langle f_{ij}, M^*f\rangle \\ &\leq D\langle M^*f, M^*f\rangle \\ &\leq D\|M^*\|^2\langle f, f\rangle.\end{aligned}$$

This completes the proof. □

In the following result we study the action of an operator on a K-woven frames.

Proposition 4.2. *Let $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ be a family of K-frames for $\mathcal{H}$. Then the following statements are equivalent:*
(i) $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is K-woven.
(ii) $\{\{Uf_{ij}\}_{j=1}^{\infty} : i \in I\}$ is UK-woven for all $U \in L(\mathcal{H})$.

Proof. (i) $\implies$ (ii) : Let $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ be a family of K-frames for $\mathcal{H}$ with universal frame bounds A and B.
Let $\{\sigma_i\}_{i\in I}$ be any partition of $\mathbb{N}$. Then for any $f \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, Uf_{ij}\rangle\langle Uf_{ij}, f\rangle &= \sum_{i\in I}\sum_{j\in\sigma_i}\langle U^*f, f_{ij}\rangle\langle f_{ij}, U^*f\rangle \\ &\leq B\langle U^*f, U^*f\rangle \\ &\leq B\|U^*\|^2\langle f, f\rangle.\end{aligned}$$

Similarly, for any $f \in \mathcal{H}$ we have

$$\begin{aligned}
\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, Uf_{ij}\rangle\langle Uf_{ij}, f\rangle &= \sum_{i\in I}\sum_{j\in\sigma_i}\langle U^*f, f_{ij}\rangle\langle f_{ij}, U^*f\rangle \\
&\geq A\langle K^*U^*f, K^*U^*f\rangle \\
&\geq A\langle (UK)^*f, (UK)^*f\rangle.
\end{aligned}$$

Hence, the family $\{\{Uf_{ij}\}_{j=1}^{\infty} : i \in I\}$ is UK-woven with universal frame bounds A and $B\|U^*\|^2$.

(ii) $\implies$ (i) : The family $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is K-woven if we choose $U = I$, the identity operator on $\mathcal{H}$. $\square$

In the following example, we show that if ϕ and ψ are K-frames for $\mathcal{H}$ such that $U\phi$ and $U\psi$ are UK-woven for some $U \in L(\mathcal{H})$. Then, in general ϕ and ψ are not K-woven.

Example 4.2. *Let $\mathcal{H} = C_0$ be the set of all sequences converging to zero and let K be the orthogonal projection of H onto span$\{e_j\}_{j=2}^{\infty}$.*
Let $\phi = \{\phi_{1j}\}_{j=1}^{\infty}$ and $\psi = \{\phi_{2j}\}_{j=1}^{\infty}$ be defined as follows:

$$\begin{aligned}
\phi \equiv \{\phi_{1j}\}_{j=1}^{\infty} &= \{0, e_1, 0, e_2, 0, e_3, 0, e_4, 0, e_5, ...\} \\
\psi \equiv \{\phi_{2j}\}_{j=1}^{\infty} &= \{e_1, 0, e_2, 0, e_3, e_3, e_4, e_4, e_5, e_5, ...\}
\end{aligned}$$

where $\{e_j\}_{j=1}^{\infty}$ is the standard orthonormal basis for $\mathcal{H}$. Then, ϕ is K-frame for $\mathcal{H}$ with lower and upper frame bound 1. One can easily verify ψ is also K-frame for $\mathcal{H}$.
Let $f = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, ...\} \in \mathcal{H}$. Then $\langle f, f\rangle = \{\alpha_1\alpha_1^*, \alpha_2\alpha_2^*, \alpha_3\alpha_3^*, \alpha_4\alpha_4^*, ...\}$
Let U be the orthogonal projection of $\mathcal{H}$ onto span$\{e_j\}_{j=3}^{\infty}$. To show that $U\phi$ and $U\psi$ are UK-woven frames for $\mathcal{H}$, first we note that

$$U\phi \equiv \{U(\phi_{1j})\}_{j=1}^{\infty} = \{0, 0, 0, 0, 0, e_3, 0, e_4, 0, e_5, 0, ...\}$$
$$U\psi \equiv \{U(\phi_{2j})\}_{j=1}^{\infty} = \{0, 0, 0, 0, e_3, e_3, e_4, e_4, e_5, e_5, ...\}.$$

For any subset σ of $\mathbb{N}$ and $f \in \mathcal{H}$, we have

$$\sum_{j\in\sigma}\langle f, U\phi_{1j}\rangle\langle U\phi_{1j}, f\rangle + \sum_{j\in\sigma^c}\langle f, U\phi_{2j}\rangle\langle U\phi_{2j}, f\rangle \leq 2\sum_{j=1}^{\infty}\langle f, e_j\rangle\langle e_j, f\rangle = 2\langle f, f\rangle$$

On the other hand, let $f \in \mathcal{H}$ and we can represent it as $f = \sum_{j=1}^{\infty}\alpha_j e_j$. Thus, we have

$$\begin{aligned}
\langle (UK)^*f, (UK)^*f\rangle &= \langle U^*f, U^*f\rangle \\
&= \langle U^*(\sum_{j=1}^{\infty}\alpha_j e_j), U^*(\sum_{j=1}^{\infty}\alpha_j e_j)\rangle \\
&= \langle \sum_{j=1}^{\infty}\alpha_j U^*e_j, \sum_{j=1}^{\infty}\alpha_j U^*e_j\rangle \\
&= \langle \sum_{j=3}^{\infty}\alpha_j e_j, \sum_{j=3}^{\infty}\alpha_j e_j\rangle \\
&= \sum_{j=3}^{\infty}\langle f, e_j\rangle\langle e_j, f\rangle \\
&\leq \sum_{j\in\sigma}\langle f, U\phi_{1j}\rangle\langle U\phi_{1j}, f\rangle + \sum_{j\in\sigma^c}\langle f, U\phi_{2j}\rangle\langle U\phi_{2j}, f\rangle
\end{aligned}$$

Hence, $U\phi$ and $U\psi$ are UK-woven frames with universal lower and upper frame bounds 1 *and* 2*, respectively.*
Now to show that ϕ and ψ are not K-woven, we choose $\sigma = \mathbb{N}\backslash\{2,4\}$. Then the family $\{\phi_{1j}\}_{j\in\sigma}\bigcup\{\phi_{2j}\}_{j\in\sigma^c} = \{0,0,0,0,e_3,0,e_4,0,e_5,...\}$ is not a K-frame for $\mathcal{H}$, since for any $A > 0$, we have

$$\begin{aligned}
\sum_{j\in\sigma}\langle e_2, \phi_{1j}\rangle\langle \phi_{1j}, e_2\rangle + \sum_{j\in\sigma^c}\langle e_2, \phi_{2j}\rangle\langle \phi_{2j}, e_2\rangle &= \sum_{j\geq 3}\langle e_2, e_j\rangle\langle e_j, e_2\rangle \\
&= 0
\end{aligned}$$

So, there exist no $A > 0$ such that

$$\sum_{j\in\sigma}\langle e_2, \phi_{1j}\rangle\langle \phi_{1j}, e_2\rangle + \sum_{j\in\sigma^c}\langle e_2, \phi_{2j}\rangle\langle \phi_{2j}, e_2\rangle \geq A\langle K^*e_2, K^*e_2\rangle$$

holds. Thus, ϕ and ψ are not K-woven.

Theorem 4.5. *Let $K \in L(\mathcal{H})$ and $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ be K-woven for $\mathcal{H}$. If $T \in L(\mathcal{H})$*

with closed range such that $\overline{R(TK)}$ is orthogonally complemented and K, T commute with each other. Then $\{\{Tf_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a K-woven frame for $R(T)$.

Proof. Since T has closed range then T has Moore-Penrose inverse operator $T^{\dagger}$ such that $TT^{\dagger}T = T$ and $T^{\dagger}TT^{\dagger} = T^{\dagger}$. So $TT^{\dagger}|_{R(T)} = I_{R(T)}$ and $(TT^{\dagger})^* = I^* = I = TT^{\dagger}$. For every $f \in R(T)$, we have

$$\begin{aligned}\langle K^*f, K^*f\rangle &= \big\langle (TT^{\dagger})^*K^*f, (TT^{\dagger})^*K^*f\big\rangle \\ &= \big\langle T^{\dagger *}T^*K^*f, T^{\dagger *}T^*K^*f\big\rangle \\ &\leq \|(T^{\dagger})^*\|^2\big\langle T^*K^*f, T^*K^*f\big\rangle\end{aligned}$$

This implies that

$$\|(T^{\dagger})^*\|^{-2}\big\langle K^*f, K^*f\big\rangle \leq \big\langle T^*K^*f, T^*K^*f\big\rangle. \tag{4.12}$$

As $R(T^*K^*) \subset R(K^*T^*)$, by using Theorem 4.2, there exists some $\lambda^{'} > 0$ such that

$$\big\langle T^*K^*f, T^*K^*f\big\rangle \leq \lambda^{'}\big\langle K^*T^*f, K^*T^*f\big\rangle. \tag{4.13}$$

Since $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is K-woven with universal bound A and B, we have

$$\begin{aligned}\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, Tf_{ij}\rangle\langle Tf_{ij}, f\rangle &= \sum_{i\in I}\sum_{j\in\sigma_i}\langle T^*f, f_{ij}\rangle\langle f_{ij}, T^*f\rangle \\ &\geq A\big\langle K^*T^*f, K^*T^*f\big\rangle \\ &\geq \frac{A}{\lambda^{'}}\langle T^*K^*f, T^*K^*f\rangle \quad \text{(Using (4.13))} \\ &\geq \frac{A}{\lambda^{'}}\|(T^{\dagger})^*\|^{-2}\langle K^*f, K^*f\rangle \quad \text{(Using (4.12))}\end{aligned}$$

On the other hand by Lemma 4.3, $\{\{Tf_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven Bessel sequence. Hence, $\{\{Tf_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a K-woven frame for $R(T)$. □

We need the following Theorem to prove our next result.

Theorem 4.6. *[54] Let E be a Hilbert module, $A, B_1, B_2 \in L(E)$ and $R(B_1) + R(B_2)$ is closed. The following statements are equivalent.*

(1) $R(A) \subset R(B_1) + R(B_2)$*;*

(2) $AA^* \leq \lambda(B_1B_1^* + B_2B_2^*)$*;*

(3) There exist $X, Y \in L(E)$ *such that* $A = B_1X + B_2Y$ *.*

Theorem 4.7. *Let* $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ *and* $\{\{g_{ij}\}_{j=1}^{\infty} : i \in I\}$ *be two K-woven frame for* $\mathcal{H}$*. Let* L_1 *and* L_2 *be defined as* $L_1, L_2 \colon l^2(\mathcal{A}) \to \mathcal{H}$*,* $L_1e_{ij} = f_{ij}$ *and* $L_2e_{ij} = g_{ij}$ *and* $R(K) \subseteq R(L_1)$*,* $R(K) \subseteq R(L_2)$*, where* $\{e_{ij}\}_{j\in\sigma_i, i\in I}$ *is the standard orthonormal basis for* $l^2(\mathcal{A})$ *and* $\{\sigma_i\}_{i\in I}$ *be any partition of* $\mathbb{N}$*. If* $L_1L_2^*$ *and* $L_2L_1^*$ *are positive operators and* $R(L_1) + R(L_2)$ *is closed, then* $\{\{f_{ij} + g_{ij}\}_{j=1}^{\infty} : i \in I\}$ *is a K-woven for* $\mathcal{H}$*.*

Proof. By the hypothesis we have

$L_1e_{ij} = f_{ij}$, $L_2e_{ij} = g_{ij}$, $R(K) \subseteq R(L_1)$ and $R(K) \subseteq R(L_2)$.

So $R(K) \subseteq R(L_1) + R(L_2)$, and by Theorem 4.6 we have

$$KK^* \leq \lambda(L_1L_1^* + L_2L_2^*)$$

for some $\lambda > 0$.

Now, let $\{\sigma_i\}_{i\in I}$ be any partition of $\mathbb{N}$.

$$\begin{aligned}
&\sum_{i\in I}\sum_{j\in\sigma_i} \langle f, f_{ij} + g_{ij}\rangle\langle f_{ij} + g_{ij}, f\rangle \\
&\qquad = \sum_{i\in I}\sum_{j\in\sigma_i} \langle f, L_1e_{ij} + L_2e_{ij}\rangle\langle L_1e_{ij} + L_2e_{ij}, f\rangle \\
&\qquad = \sum_{i\in I}\sum_{j\in\sigma_i} \langle f, (L_1 + L_2)e_{ij}\rangle\langle (L_1 + L_2)e_{ij}, f\rangle \\
&\qquad = \sum_{i\in I}\sum_{j\in\sigma_i} \langle (L_1 + L_2)^*f, e_{ij}\rangle\langle e_{ij}, (L_1 + L_2)^*f\rangle \\
&\qquad = \langle (L_1 + L_2)^*f, (L_1 + L_2)^*f\rangle \\
&\qquad = \langle (L_1 + L_2)(L_1 + L_2)^*f, f\rangle \\
&\qquad = \langle (L_1 + L_2)(L_1^* + L_2^*)f, f\rangle \\
&\qquad = \langle (L_1L_1^* + L_1L_2^* + L_2L_1^* + L_2L_2^*)f, f\rangle \\
&\qquad \geq \langle L_1L_1^* + L_2L_2^*f, f\rangle \quad \text{(As } L_1L_2^* \text{ and } L_2L_1^* \text{ are positive operators)} \\
&\qquad \geq \frac{1}{\lambda}\langle KK^*f, f\rangle \\
&\qquad = \frac{1}{\lambda}\langle K^*f, K^*f\rangle.
\end{aligned}$$

For the upper bound, let $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ and $\{\{g_{ij}\}_{j=1}^{\infty} : i \in I\}$ be two woven Bessel sequences with Bessel bound B_1 and B_2. Then it is easy to see that $\{\{f_{ij} + g_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven Bessel sequence with bound $B_1 + B_2$. And hence, $\{\{f_{ij} + g_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a K-woven for $\mathcal{H}$. □

Theorem 4.8. *For $i \in I$, let $F_i = \{f_{ij}\}_{j=1}^{\infty}$ be a family of K-frames for $\mathcal{H}$ with bounds A_i and B_i. For any $\sigma \subset \mathbb{N}$ and a fix $t \in I$, let $P_i^{\sigma}(f) = \sum_{j\in\sigma}\langle f, f_{ij}\rangle f_{ij} - \sum_{j\in\sigma}\langle f, f_{tj}\rangle f_{tj}$ for $i \neq t$. If P_i^{σ} is a positive linear operator, then the family of K-frames $\{F_i\}_{i\in I}$ is K-woven.*

Proof. Let $\{\sigma_i\}_{i\in I}$ be any partition of $\mathbb{N}$. Then, for every $f \in \mathcal{H}$, a fix $t \in I$ and $j \in \sigma_i$, we have

$$\begin{aligned}
\sum_{j\in\sigma_i}\langle f, f_{tj}\rangle\langle f_{tj}, f\rangle &= \Big\langle \sum_{j\in\sigma_i}\langle f, f_{tj}\rangle f_{tj}, f\Big\rangle \\
&= \Big\langle \sum_{j\in\sigma_i}\langle f, f_{ij}\rangle f_{ij} - P_i^{\sigma}(f), f\Big\rangle \\
&\leq \Big\langle \sum_{j\in\sigma_i}\langle f, f_{ij}\rangle f_{ij}, f\Big\rangle \quad \text{(As } P_i^{\sigma} \text{ is a positive linear operator)} \\
&= \sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle. \qquad (4.14)
\end{aligned}$$

Now,

$$\begin{aligned}
&A_t\langle K^*f, K^*f\rangle \\
&\quad\leq \sum_{j\in J}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle \\
&\quad= \sum_{j\in\sigma_1}\langle f, f_{tj}\rangle\langle f_{tj}, f\rangle + ... + \sum_{j\in\sigma_i}\langle f, f_{tj}\rangle\langle f_{tj}, f\rangle + ... + \sum_{j\in\sigma_m}\langle f, f_{tj}\rangle\langle f_{tj}, f\rangle \\
&\quad\leq \sum_{j\in\sigma_1}\langle f, f_{1j}\rangle\langle f_{1j}, f\rangle + ... + \sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle + ... + \sum_{j\in\sigma_m}\langle f, f_{mj}\rangle\langle f_{mj}, f\rangle \quad \text{(Using (4.14))} \\
&\quad\leq (B_1 + ... + B_i + ... + B_m)\langle f, f\rangle \\
&\quad= \sum_{i\in I} B_i\langle f, f\rangle
\end{aligned}$$

which implies

$$A_t\langle K^*f, K^*f\rangle \leq \sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle \leq \sum_{i\in I} B_i\langle f, f\rangle.$$

□

4.3 Conclusions

As K-frames and standard frames differ in many aspects, we introduced the concept of weaving K-frames and an atomic system for weaving K-frames in Hilbert C^*-module. In this chapter, we studied weaving K-frames from an operator theoretic point of view. We gave the equivalent definition for weaving K-frames and characterized weaving K-frames in terms of bounded linear operators. We also investigated the invariance of the woven Bessel sequence under an adjointable operator.

CHAPTER 5

Controlled K-frames in Hilbert C^*-modules

This chapter introduces the notion of controlled K-frame in Hilbert C^*-modules. We establish the equivalent condition for a controlled K-frame. We investigate some operator theoretic characterizations of controlled K-frames and controlled Bessel sequences. Moreover, we establish the relationship between the K-frames and controlled K-frames. We also investigate the invariance of a C-controlled K-frame under a suitable map T. In the end, we prove a perturbation result for controlled K-frame.

5.1 Introduction and Preliminaries

In 2014, Najati *et al.* [49] introduced the concepts of an atomic system for operators and K-frames in Hilbert C^*-modules. Controlled frames have been the subject of interest because of their ability to improve the numerical efficiency of iterative algorithms for inverting the frame operator. In 2017, Rashidi and Rahimi [51] introduced controlled frames in Hilbert C^*-modules.
We recall some basic definitions from the literature.

Definition 5.1. *[49] A sequence $\{\psi_j\}_{j\in J}$ of elements in a Hilbert $\mathcal{A}$-module $\mathcal{H}$ is said to be a K-frame ($K \in L(\mathcal{H})$) if there exist constants $C, D > 0$ such that*

$$C\langle K^*f, K^*f\rangle \leq \sum_{j\in J}\langle f, \psi_j\rangle\langle \psi_j, f\rangle \leq D\langle f, f\rangle,\ \forall\ f \in \mathcal{H}. \tag{5.1}$$

Definition 5.2. *[51] Let $\mathcal{H}$ be a Hilbert C^*-module and $C \in GL(\mathcal{H})$. A frame controlled by the operator C or C-controlled frame in Hilbert C^*-module $\mathcal{H}$ is a family of vectors*

$\{\psi_j\}_{j\in J}$, *such that there exist two constants* $A, B > 0$ *satisfying*

$$A\langle f, f\rangle \leq \sum_{j\in J}\langle f, \psi_j\rangle\langle C\psi_j, f\rangle \leq B\langle f, f\rangle, \ \forall\, f \in \mathcal{H}.$$

Likewise, $\{\psi_j\}_{j\in J}$ *is called a C-controlled Bessel sequence with bound B, if there exists* $B > 0$ *such that*

$$\sum_{j\in J}\langle f, \psi_j\rangle\langle C\psi_j, f\rangle \leq B\langle f, f\rangle, \ \forall\, f \in \mathcal{H},$$

where the sum in the above inequalities converges in norm.
If $A = B$*, we call* $\{\psi_j\}_{j\in J}$ *as C-controlled tight frame, and if* $A = B = 1$ *it is called a C-controlled Parseval frame.*

5.2 Main Results

We define below the controlled operator frame or C-controlled K-frame on a Hilbert C^*-module $\mathcal{H}$.

Definition 5.3. *Let* $\mathcal{H}$ *be a Hilbert* $\mathcal{A}$*-module over a unital* C^**-algebra,* $C \in GL^+(\mathcal{H})$ *and* $K \in L(\mathcal{H})$*. A sequence* $\{\psi_j\}_{j\in J}$ *in* $\mathcal{H}$ *is said to be a C-controlled K-frame if there exist two constants* $0 < A \leq B < \infty$ *such that*

$$A\langle C^{\frac{1}{2}}K^* f, C^{\frac{1}{2}}K^* f\rangle \leq \sum_{j\in J}\langle f, \psi_j\rangle\langle C\psi_j, f\rangle \leq B\langle f, f\rangle, \ \forall\, f \in \mathcal{H}. \tag{5.2}$$

If $C = I$, the C-controlled K-frame $\{\psi_j\}_{j\in J}$ is simply K-frame in $\mathcal{H}$ which was discussed in [49]. The sequence $\{\psi_j\}_{j\in J}$ is called a C-controlled Bessel sequence with bound B, if there exists $B > 0$ such that

$$\sum_{j\in J}\langle f, \psi_j\rangle\langle C\psi_j, f\rangle \leq B\langle f, f\rangle, \ \forall\, f \in \mathcal{H}, \tag{5.3}$$

where the sum in the above inequalities converges in norm.
We now give an example of C-controlled K-frame in Hilbert C^*-module.

Example 5.1. *Let $\mathcal{H} = C_0$ be the set of all sequences converging to zero and $\{e_j\}_{j=1}^{\infty}$ be the standard orthonormal basis for $\mathcal{H}$.*
For any $u = \{u_j\}_{j=1}^{\infty} \in \mathcal{H}$ and $v = \{v_j\}_{j=1}^{\infty} \in \mathcal{H}$

$$\langle u, v \rangle = uv^* = \{u_j v_j^*\}_{j=1}^{\infty}.$$

We define $\{\psi_j\}_{j\in J}$ as follows:

$$\{\psi_j\}_{j\in J} = \{0, 0, e_3, e_4, e_5, ...\}.$$

Let K be the orthogonal projection from $\mathcal{H}$ onto $\overline{span}\{e_j\}_{j=3}^{\infty}$ and $C \in GL^+(\mathcal{H})$ be such that

$$C(e_i) = \begin{cases} e_1 + e_2, & i = 1 \\ e_i, & otherwise \end{cases}$$

Let $f = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, ...\} \in \mathcal{H}$. Then $\langle f, f \rangle = \{\alpha_1\alpha_1^*, \alpha_2\alpha_2^*, \alpha_3\alpha_3^*, \alpha_4\alpha_4^*, ...\}$.
Now, for the upper bound, we have

$$\begin{aligned}
\sum_{j\in J} \langle f, \psi_j \rangle \langle C\psi_j, f \rangle &= \langle f, e_3 \rangle \langle C(e_3), f \rangle + \langle f, e_4 \rangle \langle C(e_4), f \rangle + \langle f, e_5 \rangle \langle C(e_5), f \rangle + ... \\
&= \langle f, e_3 \rangle \langle e_3, f \rangle + \langle f, e_4 \rangle \langle e_4, f \rangle + \langle f, e_5 \rangle \langle e_5, f \rangle + ... \\
&\leq \sum_{j\in J} \langle f, e_j \rangle \langle e_j, f \rangle \\
&= \langle f, f \rangle.
\end{aligned}$$

On the other hand, f can be written as $f = \sum_{j=1}^{\infty} \alpha_j e_j$. Thus, we have

$$\begin{aligned}
\langle C^{\frac{1}{2}}K^*f, C^{\frac{1}{2}}K^*f\rangle &= \langle CK^*f, K^*f\rangle \\
&= \big\langle CK^*(\sum_{j=1}^{\infty} \alpha_j e_j), K^*(\sum_{j=1}^{\infty} \alpha_j e_j)\big\rangle \\
&= \big\langle C(\sum_{j=3}^{\infty} \alpha_j e_j), \sum_{j=3}^{\infty} \alpha_j e_j\big\rangle \\
&= \big\langle \sum_{j=3}^{\infty} \alpha_j e_j, \sum_{j=3}^{\infty} \alpha_j e_j\big\rangle \\
&= \sum_{j=3}^{\infty} \langle f, e_j\rangle\langle e_j, f\rangle \\
&\leq \sum_{j\in J} \langle f, \psi_j\rangle\langle C\psi_j, f\rangle.
\end{aligned}$$

Hence $\{\psi_j\}_{j\in J}$ is a C-controlled K-frame with lower and upper frame bound 1.

Let $\{\psi_j\}_{j\in J}$ be a C-controlled Bessel sequence for Hilbert module $\mathcal{H}$ over $\mathcal{A}$. The operator $T\colon H \to \ell^2(\mathcal{A})$ defined by

$$Tf = \{\langle f, \psi_j\rangle\}_{j\in J},\ \forall\, f \in \mathcal{H} \tag{5.4}$$

is called the *analysis operator*. The adjoint operator $T^*\colon \ell^2(\mathcal{A}) \to \mathcal{H}$ given by

$$T^*(\{c_j\})_{j\in J} = \sum_{j\in J} c_j C\psi_j \tag{5.5}$$

is called *pre-frame operator or the synthesis operator*. By composing T and T^*, we obtain the *C-controlled frame operator $S_C\colon \mathcal{H} \to \mathcal{H}$* as

$$S_C f = T^*Tf = \sum_{j\in J} \langle f, \psi_j\rangle C\psi_j. \tag{5.6}$$

For the rest of the paper we indicate that S_C stands for the controlled frame operator as we have defined in (5.6), and S stands for the classical frame operator in Hilbert C^*-module $\mathcal{H}$ as defined in (1.9).

Lemma 5.1. *Let $C \in GL^+(\mathcal{H})$, $KC = CK$ and $R(C^{\frac{1}{2}}) \subseteq R(K^*C^{\frac{1}{2}})$ with $\overline{R((C^{\frac{1}{2}})^*)}$ is* *orthogonally complemented. Then* $\|C^{\frac{1}{2}}f\|^2 \leq \lambda^{'}\|K^*C^{\frac{1}{2}}f\|^2$ *for some* $\lambda^{'} > 0$.

Proof. Suppose $R(C^{\frac{1}{2}}) \subseteq R(K^*C^{\frac{1}{2}})$ with $\overline{R((C^{\frac{1}{2}})^*)}$ orthogonally complemented. Then by using Theorem 4.2, there exist some $\lambda^{'} > 0$ such that

$$(C^{\frac{1}{2}})(C^{\frac{1}{2}})^* \leq \lambda^{'}(K^*C^{\frac{1}{2}})(K^*C^{\frac{1}{2}})^*.$$

This implies that $\langle (C^{\frac{1}{2}})(C^{\frac{1}{2}})^*f, f\rangle \leq \lambda^{'}\langle (K^*C^{\frac{1}{2}})(K^*C^{\frac{1}{2}})^*f, f\rangle$.
Now by taking norm on both sides, we get

$$\|C^{\frac{1}{2}}f\|^2 \leq \lambda^{'}\|K^*C^{\frac{1}{2}}f\|^2.$$

□

In the following theorem, we establish an equivalence condition for C-controlled K-frame in a Hilbert C^*-module $\mathcal{H}$.

Theorem 5.1. *Let* $\mathcal{H}$ *be a finitely or countably generated Hilbert* $\mathcal{A}$ *-module over a unital* C^**-algebra* $\mathcal{A}$*,* $\{\psi_j\}_{j\in J} \subset \mathcal{H}$ *be a sequence,* $C \in GL^+(\mathcal{H})$*,* $K \in L(\mathcal{H})$*,* $KC = CK$ *and* $R(C^{\frac{1}{2}}) \subseteq R(K^*C^{\frac{1}{2}})$ *with* $\overline{R((C^{\frac{1}{2}})^*)}$ *be orthogonally complemented. Then* $\{\psi_j\}_{j\in J}$ *is a* C*-controlled* K*-frame in Hilbert* C^**-module if and only if there exist constants* $0 < A \leq B < \infty$ *such that*

$$A\|C^{\frac{1}{2}}K^*f\|^2 \leq \|\sum_{j\in J}\langle f, \psi_j\rangle\langle C\psi_j, f\rangle\| \leq B\|f\|^2,\ \forall\, f \in \mathcal{H}. \tag{5.7}$$

Proof. ($\Longrightarrow$) Obvious.
Now we assume that there exist constants $0 < A, B < \infty$ such that

$$A\|C^{\frac{1}{2}}K^*f\|^2 \leq \|\sum_{j\in J}\langle f, \psi_j\rangle\langle C\psi_j, f\rangle\| \leq B\|f\|^2,\ \forall\, f \in \mathcal{H}.$$

We prove that $\{\psi_j\}_{j\in J}$ is a C-controlled K-frame for Hilbert C^*-module $\mathcal{H}$. As S

and C are both positive operator, they are self adjoint. Thus we have

$$\begin{aligned} A\|C^{\frac{1}{2}}K^*f\|^2 &\leq \|\sum_{j\in J}\langle f,\psi_j\rangle\langle C\psi_j,f\rangle\| \\ &= \|\langle S_C f,f\rangle\| = \|\langle CSf,f\rangle\| = \|\big\langle (CS)^{\frac{1}{2}}f,(CS)^{\frac{1}{2}}f\big\rangle\|, \text{ as } S_C = CS \\ &= \|(CS)^{\frac{1}{2}}f\|^2. \end{aligned} \tag{5.8}$$

Since $R(C^{\frac{1}{2}}) \subseteq R(K^*C^{\frac{1}{2}})$ with $\overline{R((C^{\frac{1}{2}})^*)}$ is orthogonally complemented, then using Lemma 5.1, there exist some $\lambda' > 0$ such that

$$\|C^{\frac{1}{2}}f\|^2 \leq \lambda'\|K^*C^{\frac{1}{2}}f\|^2.$$

Multiplying both side by A, we get

$$\begin{aligned} A\|C^{\frac{1}{2}}f\|^2 &\leq A\lambda'\|K^*C^{\frac{1}{2}}f\|^2 \\ &\leq \lambda'\|(CS)^{\frac{1}{2}}f\|^2, \end{aligned}$$

which implies

$$\begin{aligned} &\frac{A}{\lambda'}\|C^{\frac{1}{2}}f\|^2 \leq \|S^{\frac{1}{2}}C^{\frac{1}{2}}f\|^2 \\ \Rightarrow\ &\sqrt{\frac{A}{\lambda'}}\|C^{\frac{1}{2}}f\| \leq \|S^{\frac{1}{2}}C^{\frac{1}{2}}f\|. \end{aligned} \tag{5.9}$$

Now by using Lemma 3.1, we have

$$\begin{aligned} &\langle S^{\frac{1}{2}}C^{\frac{1}{2}}f,S^{\frac{1}{2}}C^{\frac{1}{2}}f\rangle \geq \sqrt{\frac{A}{\lambda'}}\langle C^{\frac{1}{2}}f,C^{\frac{1}{2}}f\rangle \\ \Rightarrow\ &\langle C^{\frac{1}{2}}f,C^{\frac{1}{2}}f\rangle \leq \sqrt{\frac{\lambda'}{A}}\langle S_C f,f\rangle. \end{aligned}$$

Also

$$\begin{aligned} \langle C^{\frac{1}{2}}K^*f,C^{\frac{1}{2}}K^*f\rangle &\leq \|K^*\|^2\langle C^{\frac{1}{2}}f,C^{\frac{1}{2}}f\rangle \\ &\leq \|K^*\|^2\sqrt{\frac{\lambda'}{A}}\langle S_C f,f\rangle. \end{aligned}$$

This implies that

$$\frac{1}{\|K^*\|^2}\sqrt{\frac{A}{\lambda'}}\langle C^{\frac{1}{2}}K^*f, C^{\frac{1}{2}}K^*f\rangle \leq \langle S_C f, f\rangle. \tag{5.10}$$

Since S_C is positive, self adjoint and bounded $\mathcal{A}$-linear map, we can write

$$\langle S_C^{\frac{1}{2}}f, S_C^{\frac{1}{2}}f\rangle = \langle S_C f, f\rangle = \sum_{j\in J}\langle f, \psi_j\rangle\langle C\psi_j, f\rangle,$$

and hence by using Lemma 3.2, there exists some $B' > 0$ such that

$$\begin{aligned} &\langle S_C^{\frac{1}{2}}f, S_C^{\frac{1}{2}}f\rangle \leq B'\langle f, f\rangle \\ \implies\ &\langle S_C f, f\rangle \leq B'\langle f, f\rangle,\ \forall\, f \in \mathcal{H}. \end{aligned} \tag{5.11}$$

Therefore from (5.10) and (5.11), we conclude that $\{\psi_j\}_{j\in J}$ is a C-controlled K-frame in Hilbert C^*-module $\mathcal{H}$ with frame bounds $\frac{1}{\|K^*\|^2}\sqrt{\frac{A}{\lambda'}}$ and B'. □

Lemma 5.2. *Let $C \in GL^+(\mathcal{H})$, $CS_C = S_C C$ and $R(S_C^{\frac{1}{2}}) \subseteq R((CS_C)^{\frac{1}{2}})$ with $\overline{R((S_C^{\frac{1}{2}})^*)}$ is orthogonally complemented. Then $\|S_C^{\frac{1}{2}}f\|^2 \leq \lambda\|(CS_C)^{\frac{1}{2}}f\|^2$ for some $\lambda > 0$.*

Proof. By the assumption that $R(S_C^{\frac{1}{2}}) \subseteq R((CS_C)^{\frac{1}{2}})$ with $\overline{R((S_C^{\frac{1}{2}})^*)}$ orthogonally complemented. Then by using Theorem 4.2, there exists some $\lambda > 0$ such that

$$(S_C^{\frac{1}{2}})(S_C^{\frac{1}{2}})^* \leq \lambda((CS_C)^{\frac{1}{2}})((CS_C)^{\frac{1}{2}})^*.$$

This implies that

$$\begin{aligned} &\big\langle (S_C^{\frac{1}{2}})(S_C^{\frac{1}{2}})^* f, f\big\rangle \leq \lambda\big\langle ((CS_C)^{\frac{1}{2}})((CS_C)^{\frac{1}{2}})^* f, f\big\rangle \\ \Rightarrow\ &\|S_C^{\frac{1}{2}}f\|^2 \leq \lambda\|(CS_C)^{\frac{1}{2}}f\|^2,\ \forall\, f \in \mathcal{H}. \end{aligned}$$

□

In the following theorem, we prove a characterization of C-controlled Bessel sequence.

Theorem 5.2. *Let $\{\psi_j\}_{j\in J}$ be a sequence of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Suppose that C commutes with the controlled frame operator S_C and $R(S_C^{\frac{1}{2}}) \subseteq R((CS_C)^{\frac{1}{2}})$ with $\overline{R((S_C^{\frac{1}{2}})^*)}$ is orthogonally complemented. Then $\{\psi_j\}_{j\in J}$ is a C-controlled Bessel sequence with bound B if and only if the operator $U\colon \ell^2(\mathcal{A}) \to \mathcal{H}$ defined by*

$$U\{a_j\}_{j\in J} = \sum_{j\in J} a_j C\psi_j$$

is a well defined bounded operator from $\ell^2(\mathcal{A})$ into $\mathcal{H}$ with $\|U\| \leq \sqrt{B}\|C^{\frac{1}{2}}\|$.

Proof. Suppose that $\{\psi_j\}_{j\in J}$ is a C-controlled Bessel sequence with bound B. Therefore we have

$$\|\sum_{j\in J}\langle f,\psi_j\rangle\langle C\psi_j, f\rangle\| = \|\langle S_C f, f\rangle\| \leq B\|f\|^2,\ \forall\, f \in \mathcal{H}.$$

We first show that U is a well-defined operator. Let $a = \{a_j\}_{j\in J}$ and for arbitrary

$n > m$, we have

$$\begin{aligned}
\|\sum_{j=1}^{n} a_j C\psi_j - \sum_{j=1}^{m} a_j C\psi_j\|^2 &= \|\sum_{j=m+1}^{n} a_j C\psi_j\|^2 \\
&= \sup_{\|f\|=1} \big\|\langle \sum_{j=m+1}^{n} a_j C\psi_j, f\rangle\big\|^2 \\
&= \sup_{\|f\|=1} \big\|\sum_{j=m+1}^{n} a_j \langle C\psi_j, f\rangle\big\|^2 \\
&\leq \sup_{\|f\|=1} \big\|\sum_{j=m+1}^{n} \langle f, C\psi_j\rangle\langle C\psi_j, f\rangle\big\|\big\|\sum_{j=m+1}^{n} a_j a_j^*\big\| \\
&= \sup_{\|f\|=1} \big\|\langle \sum_{j=m+1}^{n} \langle f, C\psi_j\rangle C\psi_j, f\rangle\big\|\big\|\sum_{j=m+1}^{n} a_j a_j^*\big\| \\
&\leq \sup_{\|f\|=1} \big\|\langle CS_C f, f\rangle\big\|\big\|\sum_{j=m+1}^{n} a_j a_j^*\big\| \\
&= \sup_{\|f\|=1} \big\|\langle (CS_C)^{\frac{1}{2}} f, (CS_C)^{\frac{1}{2}} f\rangle\big\|\big\|\sum_{j=m+1}^{n} a_j a_j^*\big\| \\
&\leq \sup_{\|f\|=1} \|(CS_C)^{\frac{1}{2}} f\|^2 \|a\|^2 \\
&\leq \sup_{\|f\|=1} \|C^{\frac{1}{2}}\|^2 \|S_C^{\frac{1}{2}} f\|^2 \|a\|^2 \\
&\leq \sup_{\|f\|=1} B\|f\|^2 \|C^{\frac{1}{2}}\|^2 \|a\|^2 = B\|C^{\frac{1}{2}}\|^2 \|a\|^2.
\end{aligned}$$

This shows that $\sum_{j\in J} a_j C\psi_j$ is a Cauchy sequence which is convergent in $\mathcal{H}$. Thus $U(\{a_j\}_{j\in J})$ is a well defined operator from $\ell^2(\mathcal{A})$ into $\mathcal{H}$.

For boundedness of U, we consider

$$
\begin{aligned}
\|U\{a_j\}_{j\in J}\|^2 &= \sup_{\|f\|=1} \|\langle U\{a_j\}, f\rangle\|^2 \\
&= \sup_{\|f\|=1} \Big\| \sum_{j\in J} a_j \langle C\psi_j, f\rangle \Big\|^2 \\
&\leq \sup_{\|f\|=1} \Big\| \sum_{j\in J} \langle f, C\psi_j\rangle \langle C\psi_j, f\rangle \Big\| \Big\| \sum_{j\in J} a_j a_j^* \Big\| \\
&= \sup_{\|f\|=1} \Big\| \Big\langle \sum_{j\in J} \langle f, C\psi_j\rangle C\psi_j, f \Big\rangle \Big\| \Big\| \sum_{j\in J} a_j a_j^* \Big\| \\
&= \sup_{\|f\|=1} \Big\| \langle CS_C f, f\rangle \Big\| \Big\| \sum_{j\in J} a_j a_j^* \Big\| \\
&= \sup_{\|f\|=1} \Big\| \Big\langle (CS_C)^{\frac{1}{2}} f, (CS_C)^{\frac{1}{2}} f \Big\rangle \Big\| \Big\| \sum_{j\in J} a_j a_j^* \Big\| \\
&= \sup_{\|f\|=1} \Big\| (CS_C)^{\frac{1}{2}} f \Big\|^2 \|a\|^2 \\
&\leq \sup_{\|f\|=1} \|C^{\frac{1}{2}}\|^2 \|S_C^{\frac{1}{2}} f\|^2 \|a\|^2 \\
&\leq B \|C^{\frac{1}{2}}\|^2 \|a\|^2.
\end{aligned}
$$

This implies that $\|U\| \leq \sqrt{B}\|C^{\frac{1}{2}}\|$.

Now assume that U is well defined operator from $\ell^2(\mathcal{A})$ into $\mathcal{H}$ and $\|U\| \leq \sqrt{B}\|C^{\frac{1}{2}}\|$. We now prove that $\{\psi_j\}_{j\in J}$ is a C-controlled Bessel sequence with Bessel bound B.

For arbitrary $f \in \mathcal{H}$ and $\{a_j\} \in \ell^2(\mathcal{A})$, we have

$$
\begin{aligned}
\big\langle f, U\{a_j\} \big\rangle &= \big\langle f, \sum_{j\in J} a_j C\psi_j \big\rangle \\
&= \big\langle \sum_{j\in J} a_j^* C f, \psi_j \big\rangle \\
&= \sum_{j\in J} \langle Cf, \psi_j\rangle a_j^*.
\end{aligned}
$$

Therefore we get

$$
\big\langle f, U\{a_j\} \big\rangle = \big\langle \{\langle Cf, \psi_j\rangle\}, \{a_j\} \big\rangle.
$$

This implies that U has an adjoint, and $U^*f = \{\langle Cf, \psi_j\rangle\}$. Also, $\|U\| = \|U^*\|$. So we have

$$\begin{aligned}
\|U^*f\|^2 = \|\langle U^*f, U^*f\rangle\| = \|\langle UU^*f, f\rangle\| = \|\langle CS_C f, f\rangle\| \\
= \|(CS_C)^{\frac{1}{2}} f\|^2 \\
\leq B\|C^{\frac{1}{2}}\|^2 \|f\|^2. \qquad (5.12)
\end{aligned}$$

By using Lemma 5.2, we have $\|S_C^{\frac{1}{2}} f\|^2 \leq \lambda \|(CS_C)^{\frac{1}{2}} f\|^2$ for some $\lambda > 0$. Using (5.12) we get

$$\|S_C^{\frac{1}{2}} f\|^2 \leq \lambda \|(CS_C)^{\frac{1}{2}} f\|^2 \leq \lambda B \|C^{\frac{1}{2}}\|^2 \|f\|^2.$$

Therefore $\{\psi_j\}_{j\in J}$ is a C-controlled Bessel sequence with Bessel bound $\lambda B\|C^{\frac{1}{2}}\|^2$.

□

Proposition 5.1. *Let $\{\psi_j\}_{j\in J}$ be a C-controlled K-frame in $\mathcal{H}$. Then $ACKK^*I \leq S_c \leq BI$.*

Proof. Suppose $\{\psi_j\}_{j\in J}$ is a C-controlled K-frame with bounds A and B. Then

$$\begin{aligned}
& A\langle C^{\frac{1}{2}} K^* f, C^{\frac{1}{2}} K^* f\rangle \leq \sum_{j\in J} \langle f, \psi_j\rangle \langle C\psi_j, f\rangle \leq B\langle f, f\rangle,\ \forall f \in \mathcal{H}. \\
\Rightarrow\ & A\langle CKK^* f, f\rangle \leq \langle S_C f, f\rangle \leq B\langle f, f\rangle. \\
\Rightarrow\ & ACKK^* I \leq S_C \leq BI.
\end{aligned}$$

□

Proposition 5.2. *Let $\{\psi_j\}_{j\in J}$ be a C-controlled Bessel sequence in $\mathcal{H}$ and $C \in GL^+(\mathcal{H})$. Then $\{\psi_j\}_{j\in J}$ is a C-controlled K-frame for $\mathcal{H}$, if and only if there exists $A > 0$ such that $CS \geq ACKK^*$.*

Proof. The sequence $\{\psi_j\}_{j\in J}$ is a controlled K-frame for $\mathcal{H}$ with frame bounds A, B

and frame operator S_C, if and only if

$$
\begin{aligned}
& A\langle C^{\frac{1}{2}}K^*f, C^{\frac{1}{2}}K^*f\rangle \leq \sum_{j\in J}\langle f,\psi_j\rangle\langle C\psi_j, f\rangle \leq B\langle f,f\rangle,\ \forall\ f\in\mathcal{H}. \\
& \Leftrightarrow A\langle CKK^*f,f\rangle \leq \langle S_C f,f\rangle \leq B\langle f,f\rangle. \\
& \Leftrightarrow A\langle CKK^*f,f\rangle \leq \langle CSf,f\rangle \leq B\langle f,f\rangle. \\
& \Leftrightarrow ACKK^*I \leq CS.
\end{aligned}
$$

□

In the following two propositions we establish the inter-relationship between K-frame and C-controlled K-frame.

Proposition 5.3. *Let $C \in GL^+(\mathcal{H})$, $K \in L(\mathcal{H})$, $KC = CK$, $R(C^{\frac{1}{2}}) \subseteq R(K^*C^{\frac{1}{2}})$ with $\overline{R((C^{\frac{1}{2}})^*)}$ is orthogonally complemented, and $\{\psi_j\}_{j\in J}$ be a C-controlled K-frame for $\mathcal{H}$ with lower and upper frame bounds A and B, respectively. Then $\{\psi_j\}_{j\in J}$ is a K-frame for $\mathcal{H}$ with lower and upper frame bounds $A\|C^{\frac{1}{2}}\|^{-2}$ and $B\|C^{\frac{-1}{2}}\|^2$, respectively.*

Proof. Suppose $\{\psi_j\}_{j\in J}$ is a C-controlled K-frame for $\mathcal{H}$ with bound A and B. Then by Theorem 5.1, we have

$$A\|C^{\frac{1}{2}}K^*f\|^2 \leq \|\sum_{j\in J}\langle f,\psi_j\rangle\langle C\psi_j, f\rangle\| \leq B\|f\|^2,\ \forall\ f\in\mathcal{H}.$$

Now,

$$
\begin{aligned}
A\|K^*f\|^2 &= A\|C^{\frac{-1}{2}}C^{\frac{1}{2}}K^*f\|^2 \\
&\leq A\|C^{\frac{1}{2}}\|^2\|C^{\frac{-1}{2}}K^*f\|^2 \\
&\leq \|C^{\frac{1}{2}}\|^2\|\sum_{j\in J}\langle f,\psi_j\rangle\langle\psi_j, f\rangle\|.
\end{aligned}
$$

This implies that

$$A\|C^{\frac{1}{2}}\|^{-2}\|K^*f\|^2 \leq \|\sum_{j\in J}\langle f,\psi_j\rangle\langle\psi_j, f\rangle\|$$

On the other hand for every $f \in \mathcal{H}$,

$$\begin{aligned}
\|\sum_{j\in J}\langle f,\psi_j\rangle\langle\psi_j,f\rangle\| &= \|\langle Sf,f\rangle\| \\
&= \|\langle C^{-1}CSf,f\rangle\| \\
&= \|\langle (C^{-1}CS)^{\frac{1}{2}}f,(C^{-1}CS)^{\frac{1}{2}}f\rangle\| \\
&= \|(C^{-1}CS)^{\frac{1}{2}}f\|^2 \\
&\leq \|C^{\frac{-1}{2}}\|^2\|(CS)^{\frac{1}{2}}f\|^2 \\
&= \|C^{\frac{-1}{2}}\|^2\|\langle (CS)^{\frac{1}{2}}f,(CS)^{\frac{1}{2}}f\rangle\| \\
&= \|C^{\frac{-1}{2}}\|^2\|\langle CSf,f\rangle\| \\
&\leq \|C^{\frac{-1}{2}}\|^2B\|f\|^2.
\end{aligned}$$

Therefore, $\{\psi_j\}_{j\in J}$ is a K-frame with lower and upper frame bounds $A\|C^{\frac{1}{2}}\|^{-2}$ and $B\|C^{\frac{-1}{2}}\|^2$, respectively. $\square$

Proposition 5.4. *Let $C \in GL^+(\mathcal{H})$, $K \in L(\mathcal{H})$, $KC = CK$, $R(C^{\frac{1}{2}}) \subseteq R(K^*C^{\frac{1}{2}})$ with $\overline{R((C^{\frac{1}{2}})^*)}$ is orthogonally complemented. Let $\{\psi_j\}_{j\in J}$ be a K-frame for $\mathcal{H}$ with lower and upper frame bounds A and B, respectively. Then $\{\psi_j\}_{j\in J}$ is a C-controlled K-frame for $\mathcal{H}$ with lower and upper frame bounds A and $\|C\|\|S\|$, respectively.*

Proof. Suppose $\{\psi_j\}_{j\in J}$ is a K-frame with frame bounds A and B. Then by equivalence condition [33] of K-frame, we have

$$A\|K^*f\|^2 \leq \|\sum_{j\in J}\langle f,\psi_j\rangle\langle\psi_j,f\rangle\| \leq B\|f\|^2,\ \forall\ f \in \mathcal{H}.$$

For any $f \in \mathcal{H}$,

$$\begin{aligned}
A\|C^{\frac{1}{2}}K^*f\|^2 &= A\|K^*C^{\frac{1}{2}}f\|^2 \\
&\leq \|\sum_{j\in J}\langle C^{\frac{1}{2}}f,\psi_j\rangle\langle\psi_j,C^{\frac{1}{2}}f\rangle\| \\
&= \|\sum_{j\in J}\langle C^{\frac{1}{2}}f,\psi_j\rangle\psi_j,C^{\frac{1}{2}}f\rangle\| \\
&= \|\langle C^{\frac{1}{2}}Sf,C^{\frac{1}{2}}f\rangle\| \\
&= \|\langle CSf,f\rangle\|. \qquad (5.13)
\end{aligned}$$

On the other hand for every $f \in \mathcal{H}$,

$$\begin{aligned}\|\langle CSf, f\rangle\| &= \|\langle Sf, C^* f\rangle\| \\ &= \|\langle Sf, Cf\rangle\| \\ &\leq \|Sf\|\|Cf\| \\ &\leq \|C\|\|S\|\|f\|^2. \end{aligned} \tag{5.14}$$

Therefore from (5.13), (5.14) and Theorem 5.1, we conclude that $\{\psi_j\}_{j\in J}$ is a C-controlled K-frame with bounds A and $\|C\|\|S\|$. □

Theorem 5.3. *Let $C \in GL^+(\mathcal{H})$, $\{\psi_j\}_{j\in J}$ be a C-controlled K-frame for $\mathcal{H}$ with bounds A and B. Let $M, K \in L(\mathcal{H})$ with $R(M) \subset R(K)$, $\overline{R(K^*)}$ orthogonally complemented, and C commutes with M and K both. Then $\{\psi_j\}_{j\in J}$ is a C-controlled M-frame for $\mathcal{H}$.*

Proof. Suppose $\{\psi_j\}_{j\in J}$ is a C-controlled K-frame for $\mathcal{H}$ with bounds A and B. Then

$$A\langle C^{\frac{1}{2}}K^* f, C^{\frac{1}{2}}K^* f\rangle \leq \sum_{j\in J}\langle f, \psi_j\rangle\langle C\psi_j, f\rangle \leq B\langle f, f\rangle, \ \forall\, f \in \mathcal{H}. \tag{5.15}$$

Since $R(M) \subset R(K)$, from Theorem 4.2, there exists some $\lambda' > 0$ such that $MM^* \leq \lambda' KK^*$. So we have

$$\langle MM^* C^{\frac{1}{2}} f, C^{\frac{1}{2}} f\rangle \leq \lambda'\langle KK^* C^{\frac{1}{2}} f, C^{\frac{1}{2}} f\rangle.$$

Multiplying the above inequality by A, we get

$$\frac{A}{\lambda'}\langle MM^* C^{\frac{1}{2}} f, C^{\frac{1}{2}} f\rangle \leq A\langle KK^* C^{\frac{1}{2}} f, C^{\frac{1}{2}} f\rangle.$$

From (5.15), we have

$$\frac{A}{\lambda'}\langle MM^* C^{\frac{1}{2}} f, C^{\frac{1}{2}} f\rangle \leq \sum_{j\in J}\langle f, \psi_j\rangle\langle C\psi_j, f\rangle \leq B\langle f, f\rangle, \ \forall\, f \in \mathcal{H}.$$

Therefore, $\{\psi_j\}_{j\in J}$ is a C-controlled M-frame with lower and upper frame bounds $\frac{A}{\lambda'}$ and B, respectively. □

In the following result, we investigate the invariance of a C-controlled Bessel sequence under an adjointable operator.

Proposition 5.5. *Let $\{\psi_j\}_{j\in J}$ be a C-controlled Bessel sequence with bound D. Let $T \in L(\mathcal{H})$ and $CT = TC$. Then $\{T\psi_j\}_{j\in J}$ is also C-controlled Bessel sequence with bound $D\|T^*\|^2$.*

Proof. Suppose $\{\psi_j\}_{j\in J}$ is a C-controlled Bessel sequence with bound D. Then we have

$$\sum_{j\in J}\langle f,\psi_j\rangle\langle C\psi_j,f\rangle \leq D\langle f,f\rangle,\ \forall\, f\in\mathcal{H}.$$

For every $f \in \mathcal{H}$,

$$\begin{aligned}\sum_{j\in J}\langle f,T\psi_j\rangle\langle CT\psi_j,f\rangle &= \sum_{j\in J}\langle T^*f,\psi_j\rangle\langle TC\psi_j,f\rangle\\ &= \sum_{j\in J}\langle T^*f,\psi_j\rangle\langle C\psi_j,T^*f\rangle\\ &\leq D\langle T^*f,T^*f\rangle\\ &\leq D\|T^*\|^2\langle f,f\rangle.\end{aligned}$$

Thus $\{T\psi_j\}_{j\in J}$ is also C-controlled Bessel sequence with bound $D\|T^*\|^2$. □

Now, we investigate the invariance of a C-controlled K-frame under an adjointable operator.

Theorem 5.4. *Let $C \in GL^+(\mathcal{H})$, $K \in L(\mathcal{H})$ and $\{\psi_j\}_{j\in J}$ be a C-controlled K-frame for $\mathcal{H}$ with lower and upper bounds A and B, respectively. If $T \in L(\mathcal{H})$ with closed range such that $\overline{R(TK)}$ is orthogonally complemented and C,K,T commute with each other. Then $\{T\psi_j\}_{j\in J}$ is a C-controlled K-frame for $R(T)$.*

Proof. Suppose $\{\psi_j\}_{j\in J}$ is a C-controlled K-frame for $\mathcal{H}$ with bound A and B. Then

$$A\langle C^{\frac{1}{2}}K^*f, C^{\frac{1}{2}}K^*f\rangle \leq \sum_{j\in J}\langle f,\psi_j\rangle\langle C\psi_j,f\rangle \leq B\langle f,f\rangle, \forall\, f\in\mathcal{H}.$$

We know that if T has closed range then T has Moore-Penrose inverse $T^\dagger$ such that $TT^\dagger T = T$ and $T^\dagger TT^\dagger = T^\dagger$. So $TT^\dagger|_{R(T)} = I_{R(T)}$ and $(TT^\dagger)^* = I^* = I = TT^\dagger$.

We have

$$\begin{aligned}\left\langle K^*C^{\frac{1}{2}}f, K^*C^{\frac{1}{2}}f\right\rangle &= \left\langle (TT^{\dagger})^*K^*C^{\frac{1}{2}}f, (TT^{\dagger})^*K^*C^{\frac{1}{2}}f\right\rangle \\ &= \left\langle T^{\dagger *}T^*K^*C^{\frac{1}{2}}f, T^{\dagger *}T^*K^*C^{\frac{1}{2}}f\right\rangle \\ &\leq \|(T^{\dagger})^*\|^2\left\langle T^*K^*C^{\frac{1}{2}}f, T^*K^*C^{\frac{1}{2}}f\right\rangle.\end{aligned}$$

This implies that

$$\|(T^{\dagger})^*\|^{-2}\left\langle K^*C^{\frac{1}{2}}f, K^*C^{\frac{1}{2}}f\right\rangle \leq \left\langle T^*K^*C^{\frac{1}{2}}f, T^*K^*C^{\frac{1}{2}}f\right\rangle. \tag{5.16}$$

Since $R(T^*K^*) \subset R(K^*T^*)$, by using Theorem 4.2, there exists some $\lambda^{'} > 0$ such that

$$\left\langle T^*K^*C^{\frac{1}{2}}f, T^*K^*C^{\frac{1}{2}}f\right\rangle \leq \lambda^{'}\left\langle K^*T^*C^{\frac{1}{2}}f, K^*T^*C^{\frac{1}{2}}f\right\rangle. \tag{5.17}$$

Therefore, using (5.16) and (5.17) we get

$$\begin{aligned}\sum_{j\in J}\langle f, T\psi_j\rangle\langle CT\psi_j, f\rangle &= \sum_{j\in J}\langle T^*f, \psi_j\rangle\langle TC\psi_j, f\rangle \\ &= \sum_{j\in J}\langle T^*f, \psi_j\rangle\langle C\psi_j, T^*f\rangle \\ &\geq A\left\langle C^{\frac{1}{2}}K^*T^*f, C^{\frac{1}{2}}K^*T^*f\right\rangle \\ &\geq A\lambda^{'}\langle T^*C^{\frac{1}{2}}K^*f, T^*C^{\frac{1}{2}}K^*f\rangle \\ &\geq A\lambda^{'}\|(T^{\dagger})^*\|^{-2}\langle C^{\frac{1}{2}}K^*f, C^{\frac{1}{2}}K^*f\rangle.\end{aligned}$$

This gives the lower frame inequality for $\{T\psi_j\}_{j\in J}$. On the other hand by Proposition 5.5, $\{T\psi_j\}_{j\in J}$ is a C-controlled Bessel sequence. So $\{T\psi_j\}_{j\in J}$ is a C-controlled K-frame for $R(T)$. □

Theorem 5.5. *Let $C \in GL^+(\mathcal{H})$, $K \in L(\mathcal{H})$ and $\{\psi_j\}_{j\in J}$ be a C-controlled K-frame for $\mathcal{H}$ with lower and upper bound A, B respectively. If $T \in L(\mathcal{H})$ is a isometry such that $R(T^*K^*) \subset R(K^*T^*)$ with $\overline{R(TK)}$ is orthogonally complemented and C, K, T commute with each other. Then $\{T\psi_j\}_{j\in J}$ is a C-controlled K-frame for $\mathcal{H}$.*

Proof. By Theorem 4.2, there exist some $\lambda > 0$ such that $\|T^*K^*C^{\frac{1}{2}}f\|^2 \leq \lambda\|K^*T^*C^{\frac{1}{2}}f\|^2$.

Suppose A is a lower bound for the C-controlled K-frame $\{\psi_j\}_{j\in J}$. Since T is an isometry, then

$$\begin{aligned}
\frac{A}{\lambda}\|C^{\frac{1}{2}}K^*f\|^2 &= \frac{A}{\lambda}\|T^*C^{\frac{1}{2}}K^*f\|^2 \\
&\leq A\|K^*T^*C^{\frac{1}{2}}f\|^2 \\
&= A\|C^{\frac{1}{2}}K^*T^*f\|^2 \\
&\leq \sum_{j\in J}\langle T^*f,\psi_j\rangle\langle C\psi_j,T^*f\rangle \\
&= \sum_{j\in J}\langle f,T\psi_j\rangle\langle TC\psi_j,f\rangle \\
&= \sum_{j\in J}\langle f,T\psi_j\rangle\langle CT\psi_j,f\rangle. \qquad (5.18)
\end{aligned}$$

Therefore from Proposition 5.5 and inequality (5.18), we conclude that $\{T\psi_j\}_{j\in J}$ is a C-controlled K-frame for $\mathcal{H}$ with bounds $\dfrac{A}{\lambda}$ and $B\|T^*\|^2$. □

Now, we prove a perturbation result for C-controlled K-frame.

Theorem 5.6. *Let $F=\{f_j\}_{j\in J}$ be a C-controlled K-frame for $\mathcal{H}$, with controlled frame operator S_C. Suppose $K\in L(\mathcal{H})$, $KC=CK$, $R(C^{\frac{1}{2}})\subseteq R(K^*C^{\frac{1}{2}})$ with $\overline{R((C^{\frac{1}{2}})^*)}$ is orthogonally complemented . If $G=\{g_j\}_{j\in J}$ is a non zero sequence in $\mathcal{H}$, and $E=T_F-T_G$ be a compact operator, where $T_G(\{c_j\}_{j\in J})=\sum_{j\in J}c_jg_j$ for $\{c_j\}_{j\in J}\in\ell^2(\mathcal{A})$, then $G=\{g_j\}_{j\in J}$ is a C-controlled K-frame for $\mathcal{H}$.*

Proof. Let $\{f_j\}_{j\in J}$ be a C-controlled K-frame with bounds A and B, then because of Theorem 5.1, we have

$$A\|C^{\frac{1}{2}}K^*f\|^2\leq\|\sum_{j\in J}\langle f,f_j\rangle\langle Cf_j,f\rangle\|\leq B\|f\|^2,\ \forall\ f\in\mathcal{H}.$$

This implies $\|T_F\|^2\leq B\|C^{\frac{-1}{2}}\|^2$.
Let $V=T_F-E$ be an operator from $\ell^2(\mathcal{A})$ into $\mathcal{H}$. Since T_F and E are bounded, then the operator V is bounded. Therefore $\|V\|=\|V^*\|$.

For any $f \in \mathcal{H}$,

$$\begin{aligned}
V^* f &= T_F^* f - E^* f \\
&= \{\langle f, f_j \rangle\}_{j\in J} - \{\langle f, f_j - g_j \rangle\}_{j\in J} \\
&= \{\langle f, f_j \rangle\}_{j\in J} - \{\langle f_j - g_j, f \rangle^*\}_{j\in J} \\
&= \{\langle f, f_j \rangle\}_{j\in J} - \{\langle f_j, f \rangle^* - \langle g_j, f \rangle^*\}_{j\in J} \\
&= \{\langle f, f_j \rangle\}_{j\in J} - \{\langle f, f_j \rangle - \langle f, g_j \rangle\}_{j\in J} \\
&= \{\langle f, g_j \rangle\}_{j\in J}.
\end{aligned}$$

We have

$$V(\{c_j\}_{j\in J}) = \sum_{j\in J} c_j g_j, \text{ and } S_G = VV^*. \tag{5.19}$$

Now using (5.19), we have

$$\begin{aligned}
\|\langle f, CS_G f \rangle\| = \|\langle f, CVV^* f \rangle\| &= \|\langle C^{\frac{1}{2}} V f, C^{\frac{1}{2}} V f \rangle\| \\
&= \|C^{\frac{1}{2}} V f\|^2 \\
&\leq \|C^{\frac{1}{2}}\|^2 \|V f\|^2 \\
&= \|C^{\frac{1}{2}}\|^2 \|(T_F - E) f\|^2 \\
&\leq \|C^{\frac{1}{2}}\|^2 \|T_F - E\|^2 \|f\|^2 \\
&\leq (\|T_F\|^2 + 2\|T_F\|\|E\| + \|E\|^2) \|C^{\frac{1}{2}}\|^2 \|f\|^2 \\
&\leq \left(B\|C^{\frac{-1}{2}}\|^2 + 2\sqrt{B}\|C^{\frac{-1}{2}}\|\|E\| + \|E\|^2\right) \|C^{\frac{1}{2}}\|^2 \|f\|^2 \\
&= B\Big(\|C^{\frac{-1}{2}}\| + \frac{\|E\|}{\sqrt{B}}\Big)^2 \|C^{\frac{1}{2}}\|^2 \|f\|^2. \qquad (5.20)
\end{aligned}$$

This inequality shows that $\{g_j\}_{j\in J}$ is a controlled Bessel sequence with bound $B\Big(\|C^{\frac{-1}{2}}\| + \frac{\|E\|}{\sqrt{B}}\Big)^2 \|C^{\frac{1}{2}}\|^2$.

Again we have

$$\begin{aligned} VV^* &= (T_F - E)(T_F - E)^* \\ &= (T_F - E)(T_F^* - E^*) \\ &= T_F T_F^* - T_F E^* - E T_F^* + EE^* \\ &= S_F - T_F E^* - E T_F^* + EE^*. \end{aligned}$$

Since E, T_F and S_F are compact operators, then $S_F - T_F E^* - E T_F^* + EE^*$ is a compact operator. Therefore $S_F - T_F E^* - E T_F^* + EE^* + I$ is a bounded operator with closed range. Thus, $VV^* = S_F - T_F E^* - E T_F^* + EE^*$ is a bounded operator with closed range. Clearly, V and its adjoint operator $V^* f = \{\langle f, g_j \rangle\}_{j \in J}$ are injective. This implies VV^* is injective as composition of two injective operator is injective. Hence $VV^*(= S_G)$ is bounded below. So there exists some constant $A > 0$ such that

$$A \|C^{\frac{1}{2}} f\| \leq \|S_G C^{\frac{1}{2}} f\|. \tag{5.21}$$

Now

$$\begin{aligned} \|C^{\frac{1}{2}} K^* f\|^2 &= \|K^* C^{\frac{1}{2}} f\|^2 \\ &\leq \|K^*\|^2 \|C^{\frac{1}{2}} f\|^2 \\ &\leq \frac{1}{A^2} \|K^*\|^2 \|S_G C^{\frac{1}{2}} f\|^2. \end{aligned}$$

This implies that

$$\frac{A^2}{\|K^*\|^2} \|C^{\frac{1}{2}} K^* f\|^2 \leq \|S_G C^{\frac{1}{2}} f\|^2. \tag{5.22}$$

Therefore from (5.20) and (5.22), we conclude that $G = \{g_j\}_{j \in J}$ is a C-controlled K-frame for $\mathcal{H}$ with frame bounds $\frac{A^2}{\|K^*\|^2}$ and $B\Big(\|C^{\frac{-1}{2}}\| + \frac{\|E\|}{\sqrt{B}}\Big)^2 \|C^{\frac{1}{2}}\|^2$. $\square$

5.3 Conclusions

In this chapter, we introduced the concept of controlled K-frame in Hilbert C^*-modules. We established the equivalent condition for a controlled K-frame as it is much easier to work. We investigated some operator theoretic characterizations of controlled K-frames and controlled Bessel sequences. We also established the relationship between the K-frames and controlled K-frames. We studied the invariance of a C-controlled K-frame under a suitable map T. Finally, we proved a perturbation result for controlled K-frame.

CHAPTER 6

Conclusions and Future Work

In this thesis, we defined and studied regular k-distance set as well as regular k-distance frame, in particular, regular k-distance tight frames in Hilbert space. We introduced the definition of dual frames for a regular k-distance set. We require the notion of a dual frame to reconstruct a vector from its frame coefficients. We established Perturbation theorems for various notions of frames in Hilbert space and in Hilbert C^*-modules as they are essential and valuable tools to construct new frames close to the given one. We studied and characterized the different frames in Hilbert C^*-modules from an operator theoretic point of view. We generalized the notions of frame theory in Hilbert space into Hilbert C^*-modules and showed that the results share many beneficial properties with their corresponding notions in a Hilbert space. We established an equivalent definition for the notions introduced in Hilbert C^*-modules. An equivalent definition is much easier to be applied, as well as permits us to study the various types of frames from the operator theory point of view.

From the analysis of the work presented in this thesis, there are several opportunities for future research, which are mentioned below:

1. The broad area of applications signifies a large prospective of problems for the investigation.

2. One can study the dynamical properties of the frame.

3. One can study the perturbation result for a regular k-distance frame in Hilbert space with lesser conditions.

4. Many portions of the area are still not explored by the community.

Publications

1. **Ekta Rajput**, Nabin Kumar Sahu, and Vishnu Narayan Mishra, "Woven g-frames in Hilbert C^*-modules", Korean J. Math. 29 (2021), No. 1, pp. 41–55.

2. Nabin K. Sahu, **Ekta Rajput**, "An Insight into the Frames in Hilbert C^*-modules", ICMAC 2019: Mathematical Analysis and Computing, pp. 581-601.

3. **Ekta Rajput**, N. K. Sahu, and Vishnu Narayan Mishra, "Controlled K-frames in Hilbert C^*-modules", Korean J. Math. 30 (2022), No. 1, pp. 91–107.

4. **Ekta Rajput** and N. K. Sahu, "Representation of frames as regular k-distance sets", J. Pseudo-Differ. Oper. Appl. 13, 58 (2022).

5. **Ekta Rajput**, N. K. Sahu and R. N. Mohapatra, "Weaving K-frames in Hilbert C^*-modules". (Under review)

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